Università di Roma Tor Vergata
Engineering Sciences

Mathematical analysis of some exact unsteady solutions of the Navier-Stokes Equations

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Introduction

The Navier-Stokes equations changed the world!

They are introduced among the 17 more important equations of all history

17 Equations That Changed the World
by Ian Stewart

1.	**Pythagoras's Theorem**	$a^2 + b^2 = c^2$	Pythagoras,530 BC
2.	**Logarithms**	$\log xy = \log x + \log y$	John Napier, 1610
3.	**Calculus**	$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$	Newton, 1668
4.	**Law of Gravity**	$F = G\frac{m_1 m_2}{r^2}$	Newton, 1687
5.	**The Square Root of Minus One**	$i^2 = -1$	Euler, 1750
6.	**Euler's Formula for Polyhedra**	$V - E + F = 2$	Euler, 1751
7.	**Normal Distribution**	$\Phi(x) = \frac{1}{\sqrt{2\pi\rho}} e^{\frac{(x-\mu)^2}{2\rho^2}}$	C.F. Gauss, 1810
8.	**Wave Equation**	$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	J. d'Almbert, 1746
9.	**Fourier Transform**	$f(\omega) = \int_{\infty}^{\infty} f(x) e^{-2\pi i x \omega} \mathrm{d}x$	J. Fourier, 1822
10.	**Navier-Stokes Equation**	$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \nabla \cdot \mathbf{T} + \mathbf{f}$	C. Navier, G. Stokes, 1845
11.	**Maxwell's Equations**	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ $\nabla \cdot \mathbf{H} = 0$ $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$ $\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial E}{\partial t}$	J.C. Maxwell, 1865
12.	**Second Law of Thermodynamics**	$\mathrm{d}S \geq 0$	L. Boltzmann, 1874
13.	**Relativity**	$E = mc^2$	Einstein, 1905
14.	**Schrodinger's Equation**	$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi$	E. Schrodinger, 1927
15.	**Information Theory**	$H = -\sum p(x) \log p(x)$	C. Shannon, 1949
16.	**Chaos Theory**	$x_{t+1} = kx_t(1 - x_t)$	Robert May, 1975
17.	**Black-Scholes Equation**	$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0$	F. Black, M. Scholes, 1990

Historical Background

Claude-Louis Navier (1822)

His approach was based on:

- Molecular view of the fluid
- Incompressible fluids

Sir George G. Stokes (1845)

He indipendently derived the equations by assuming:

- Continuum hypothesis model
- Viscous action

Nowadays

- Still one of the seven unsolved problems of modern mathematics

- The Clay Mathematics Institute will award anyone who will solve it with a 1 million dollar prize

Governing equation of a fluid flow

Incompressible fluids

- Conservation of Mass

$$\frac{DM}{Dt} = 0$$

- Balance of momentum

$$\frac{D\vec{P}}{Dt} = \sum \vec{F}$$

4 unknowns → 4 equations

Compressible fluids

- Conservation of Mass

$$\frac{DM}{Dt} = 0$$

- Balance of Momentum

$$\frac{D\vec{P}}{Dt} = \sum \vec{F}$$

- Conservation of Energy

$$\frac{DE}{Dt} = \dot{Q} + \dot{L}$$

- Equation of state

Derivation of the Equation

From Reynolds transport theorem:

$$\frac{D\vec{P}}{Dt} = \int_V \frac{\partial(\rho\vec{u})}{\partial t} dV + \int_S \rho\vec{u}(\vec{u}\cdot\hat{n})dS = \vec{F}_B + \vec{F}_S$$

- *Body forces* $\vec{F}_B = \int_V \rho\vec{f} dV$
- *Surface forces* $\vec{F}_S = \int_S \underline{\Sigma}\cdot\hat{n} dS$

Applying the Divergence Theorem we obtain:

$$\frac{\partial\rho\vec{u}}{\partial t} + \nabla\cdot(\rho\vec{u}\vec{u}) = \rho\vec{f} + \nabla\cdot\underline{\Sigma}$$

Where $\underline{\Sigma}$ is the stress tensor composed of an isotropic and deviatoric part

$$\underline{\Sigma} = -p\underline{I} + \underline{\tau}$$

Substituting the Constitutive model for a Newtonian fluid in the previous expression

$$\underline{\tau} = 2\mu \underline{E} - \frac{2}{3}\mu(\nabla \cdot \vec{u})\underline{I}$$

- Stress is a linear function of the rate of deformation
- Homogeneous and isotropic material
- μ is the dynamic viscosity, assumed to be space indipendent

We obtain the **Navier-Stokes equations**:

$$\rho \frac{D\vec{\mathbf{u}}}{Dt} = -\nabla \mathbf{p} + \rho \vec{\mathbf{f}} + \frac{\mu}{3}\nabla(\nabla \cdot \vec{\mathbf{u}}) + \mu \nabla^2 \vec{\mathbf{u}}$$

Under the hypothesis of an incompressible flow $\nabla \cdot \vec{u} = 0$

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \rho \vec{f} + \mu \nabla^2 \vec{u}$$

Presentation of the problem

- The Navier-Stokes equations are a set of **non-linear partial differential equations**
- Their analytical solution is not always possible

My analysis was carried out according to the following characteristics:

- ***Parallel flow***: only one velocity component different from zero. Given a velocity field $\vec{u} = \vec{u}(u, v, w)$ then $v = w = 0$
- ***Unsteady flow***: $\frac{\partial(\cdot)}{\partial t} \neq 0$
- ***Gravitational forces*** *were neglected*
- ***Laminar flow***

Cases Analysed

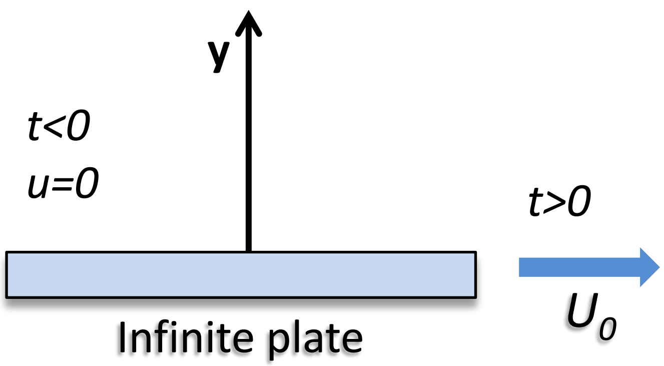
1. Stokes’s First Problem

2. Stokes’s Second Problem

3. Womersley’s Flow

Stokes's first problem

Flow near a flat plate initially at rest *suddenly accelerated* to a constant velocity U_0



Boundary Conditions

$$u(0) = \begin{cases} Uo & t > 0 \\ 0 & t < 0 \end{cases} \qquad u(y \to \infty) = 0$$

Governing equation

$$\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

The expected function is $u = U_0 f(y,t,\nu)$

- Adimensional group $\eta = \dfrac{y}{2\sqrt{\nu t}}$ → $u = U_0 f(\eta)$
- Chain rule

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = -\frac{\eta}{4t} U_0 f^{'}(\eta) \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial t}\right)^2 = \frac{1}{4\nu t} U_0 f^{''}(\eta)$$

Differential equation: $f^{''} + 2\eta f^{'} = 0$

with new boundary conditions

Strategy

noting that $\dfrac{d}{d\eta}\left(log\left(\dfrac{df}{d\eta}\right)\right) = \dfrac{f^{''}}{f'}$ we obtain $\dfrac{d}{d\eta}\left(log\left(\dfrac{df}{d\eta}\right)\right) = -2\eta$

By separation of variables:

$$f(\eta) = \int_0^{\eta} c_1 e^{-\eta^2} d\eta + c_2$$

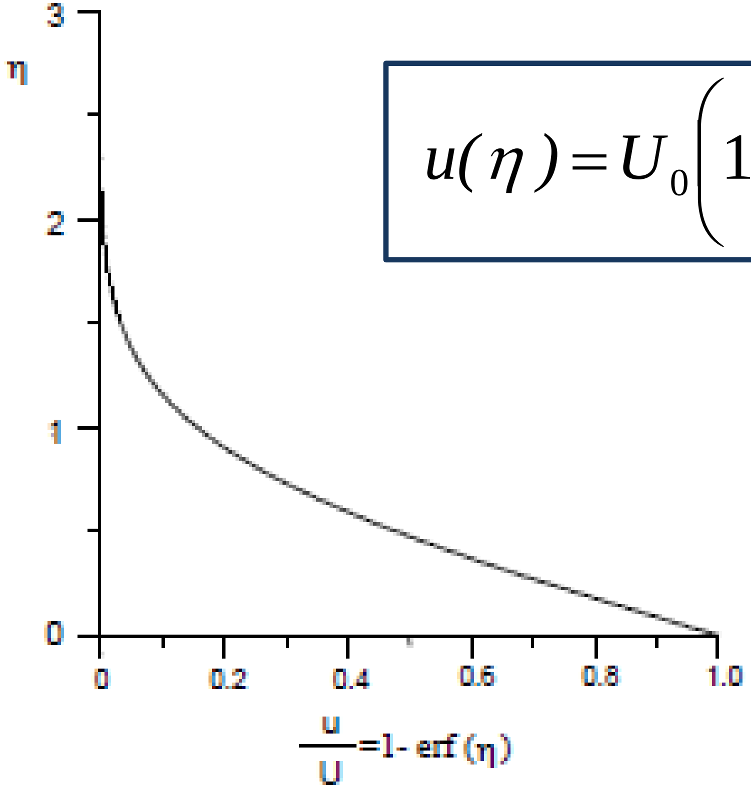
Apply boundary conditions:

$$f(0) = 1 \quad \longrightarrow \quad c_2 = 1$$

$$f(\eta \to \infty) = 0 \quad \longrightarrow \quad c_1 = -\frac{2}{\sqrt{\pi}}$$

Final Solution

$$u(\eta) = U_0\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}\, d\eta\right) = U_0 erfe(\eta)$$



- $erfe(\eta)$ is the complementary error function

$$erfe(\eta) = 1 - erf(\eta)$$

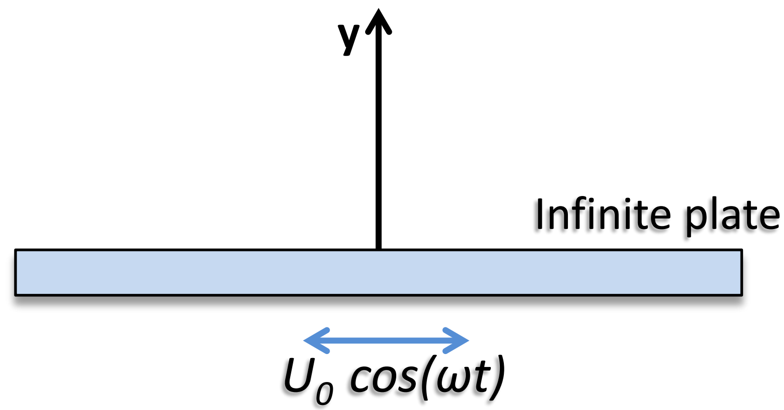
- Values can be found in tables

Stokes's second problem

Flow about an infinite plate moving with linear *harmonic oscillations*

Boundary Conditions

- $u(0,t) = U_0 \cos(\omega t)$
- $u(y \to \infty, t) = 0$



Governing equation

Same conditions as in the previous case, same starting differential equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

Expected solution: $u(y,t) = Re\left\{F(y)e^{-\iota\omega t}\right\}$

Substitutions:

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 F(y)}{\partial y^2} = U_0 F''(y) e^{-\imath\omega t}$$

$$\frac{\partial u}{\partial t} = \frac{\partial F(y)}{\partial t} = U_0 F(-\imath\omega) e^{-\imath\omega t}$$

$$F'' + \frac{i\omega}{\nu} F = 0$$

2° order differential equation

Strategy

Characteristic polynomial

$$\lambda^2 + \frac{i\omega}{\nu} = 0$$

$$\lambda = \pm\sqrt{-\frac{i\omega}{\nu}} = \pm i(1+i)\sqrt{\frac{\omega}{2\nu}} = \pm\kappa(i-1)$$

Solution

$$F(y) = c_1 e^{\kappa(i-1)y} + c_2 e^{-\kappa(i-1)y}$$

Apply the new Boundary Conditions

$$\begin{cases} F(0) = U_0 & \rightarrow \quad c_1 = U_0 \\ F(y \to \infty) = 0 & \rightarrow \quad c_2 = 0 \end{cases}$$

Substituting in our initial guess $u(y,t) = Re\left\{U_0 e^{-\kappa y} e^{(\omega t - \kappa y)i}\right\}$

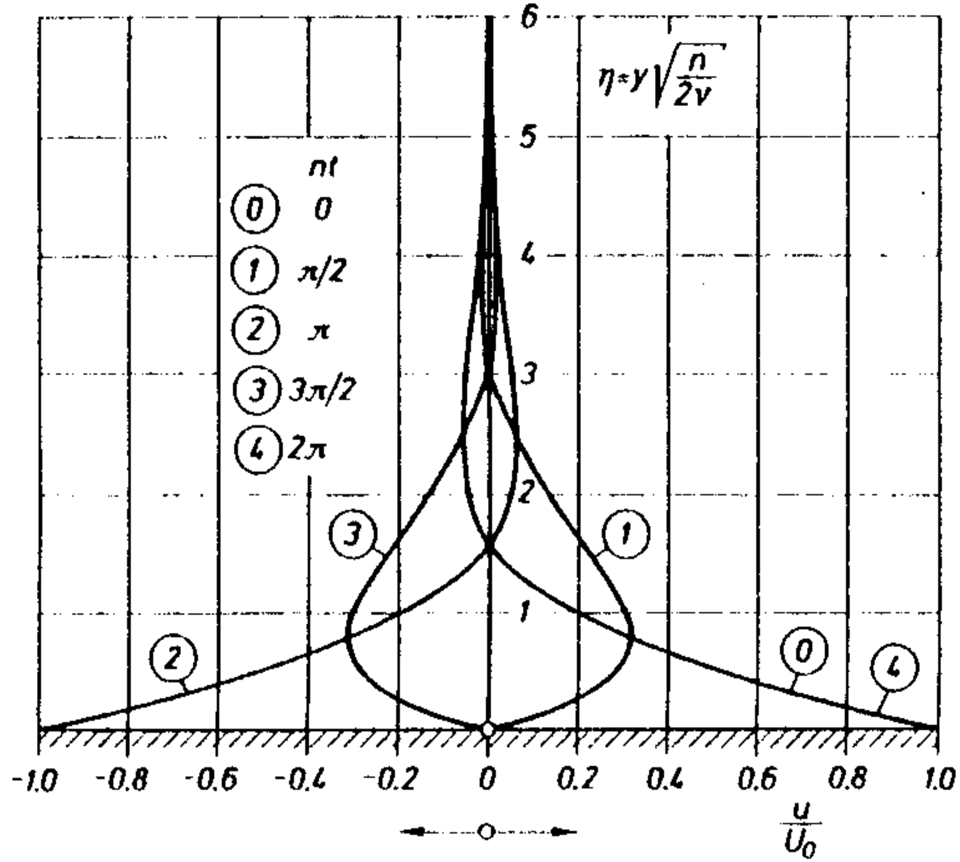
Final Solution

$$u(y,t) = U_0 e^{-\kappa y}\left[\cos(\omega t - \kappa y)\right]$$

- The velocity profile has the shape of a damped harmonic oscillation

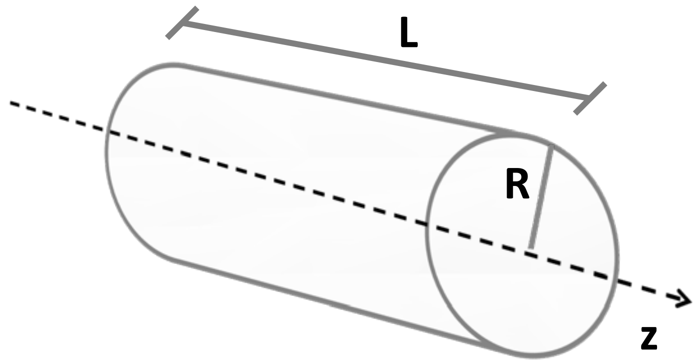
- Depth of penetration $\delta = \frac{2\pi}{\kappa}$

- Lag between a layer near the plate and one at distance y



Womersley's flow

Fluid flow in a cylinder of radius R and length L with a *periodic pressure gradient* $\frac{\partial P}{\partial z} = -\rho K e^{i\omega t}$



The Navier-Stokes equations written in the cylindrical coordinates are needed:

- Given a flow, the only remaining component is u_z

$$\vec{u} = (u_r, u_\theta, u_z)$$

$$\left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z}\right) = f_z - \frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right)$$

Governing equation

$$\frac{\partial u}{\partial t} = \mathrm{K}e^{i\omega t} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

Boundary condition

- No-slip condition: $u(r = R, t) = 0$

$$u(r,t) = V(r)e^{i\omega t}$$

Expected solution

Mathematical manipulations:

Derivatives → Multiply by → Substitutions

Derivatives

- $\frac{\partial u}{\partial t} = i\omega V(r)e^{i\omega t}$
- $\frac{\partial^2 u}{\partial r^2} = V''(r)e^{i\omega t}$

Multiply by

- $\frac{r^2}{\nu}$

Substitutions

- $x = r\sqrt{\frac{i\omega}{\nu}}$
- $\frac{dV}{dr} = \frac{dV}{dx}\frac{dx}{dr}$

Resulting Differential Equation

$$x^2V'' + xV' - x^2V = -\mathrm{K}\frac{x^2}{i\omega}$$

Change of Variable

$$V = \hat{V} + \frac{\mathrm{K}}{i\omega}$$

$$x^2\hat{V}'' + x\hat{V}' - x^2\hat{V} = 0$$

Modified Bessel Equation of Zero Order

$$u(r,t) = e^{i\omega t}\left(AI_0\left(r\sqrt{\frac{i\omega}{\nu}}\right) + \frac{\mathrm{K}}{i\omega}\right)$$

Apply B.C.

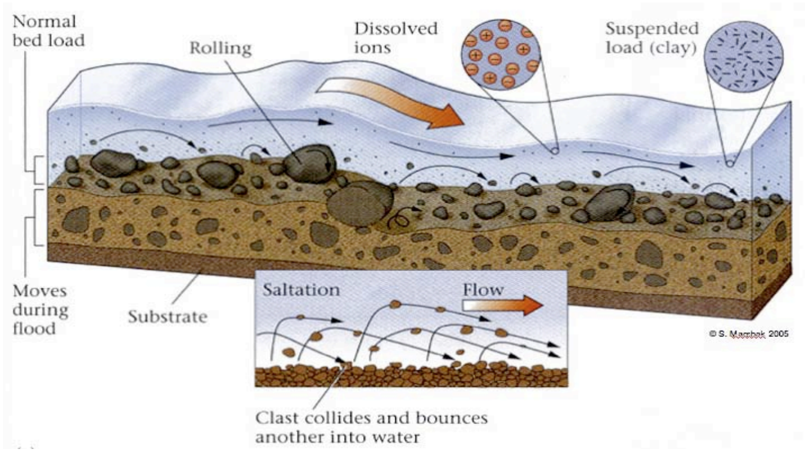
$$u(r,t) = \frac{\mathrm{K}}{i\omega}e^{i\omega t}\left[1 - \frac{I_0\left(r\sqrt{\frac{i\omega}{\nu}}\right)}{I_0\left(R\sqrt{\frac{i\omega}{\nu}}\right)}\right] = \frac{\mathrm{K}}{i\omega}e^{i\omega t}\left[1 - \frac{J_0\left(r\sqrt{\frac{-i\omega}{\nu}}\right)}{J_0\left(R\sqrt{\frac{-i\omega}{\nu}}\right)}\right]$$

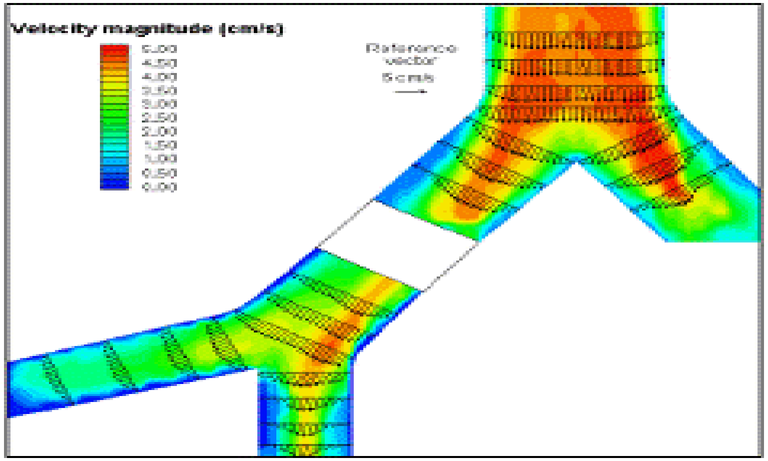
Conclusions

Engineering Applications:

Environmental flows:

- Flow over the ocean bed
- Sediment transport mechanics





Biological flows:

- Model of motion of the blood in straight arteries
- Womersley's number $Wo = r\sqrt{\frac{\omega}{\nu}}$

Thanks for the attention!