UNIVERSITA' degli STUDI di ROMA
TOR VERGATA

The Inertia Torque of the Hooke Joint

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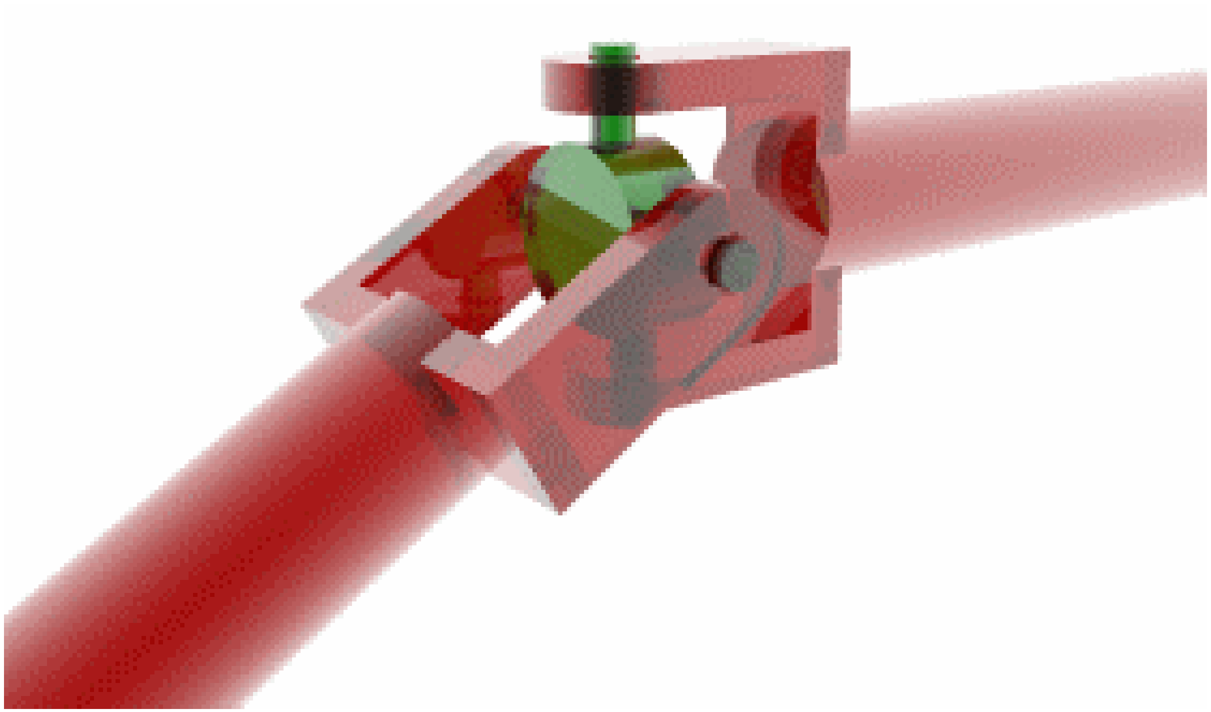
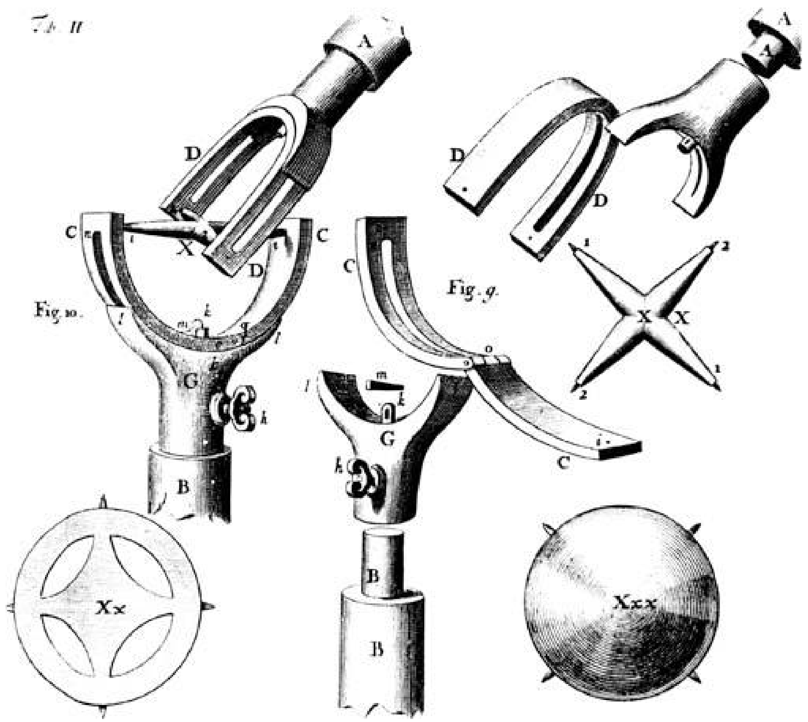


History

Already described by Philo of Byzantium in the III century B.C.

Gerolamo Cardano in the XVI century proposed it as motive power transmission joint.

Robert Hooke analysed it in his *Helioscopes* in the XVII century



Nowadays it is called in several ways: **Cardan Joint**, **Hooke Joint**, or as Hooke himself called it, **Universal Joint**

Common Applications

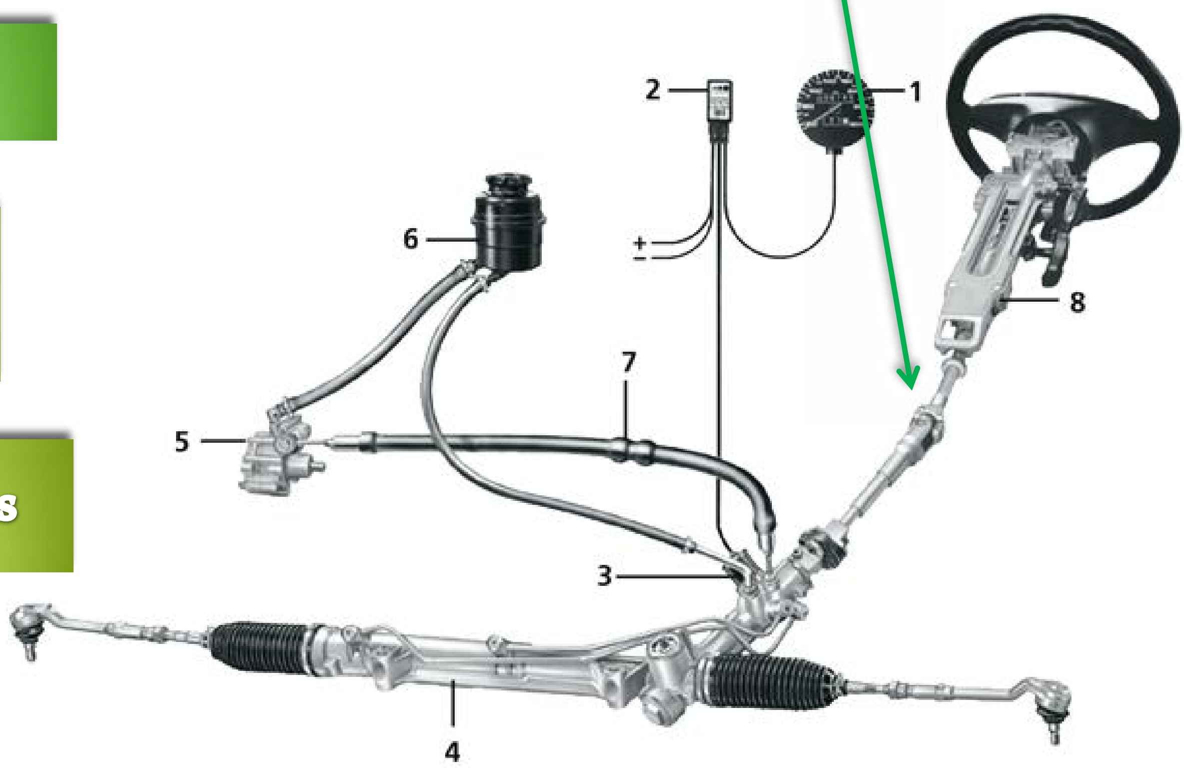
Vehicle Steering Systems

Transmission Line

Front/Rear Driveline

Heavy Duty Vehicles

Cardan Joint



Common Applications

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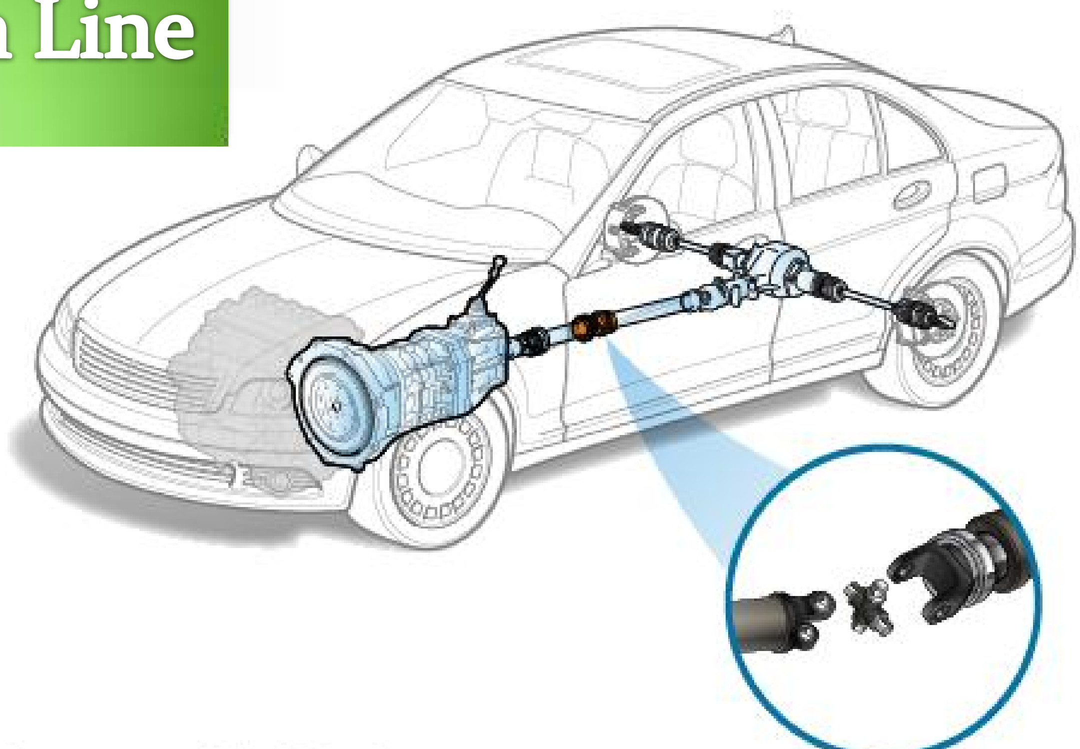


Image courtesy of ClearMechanic.com

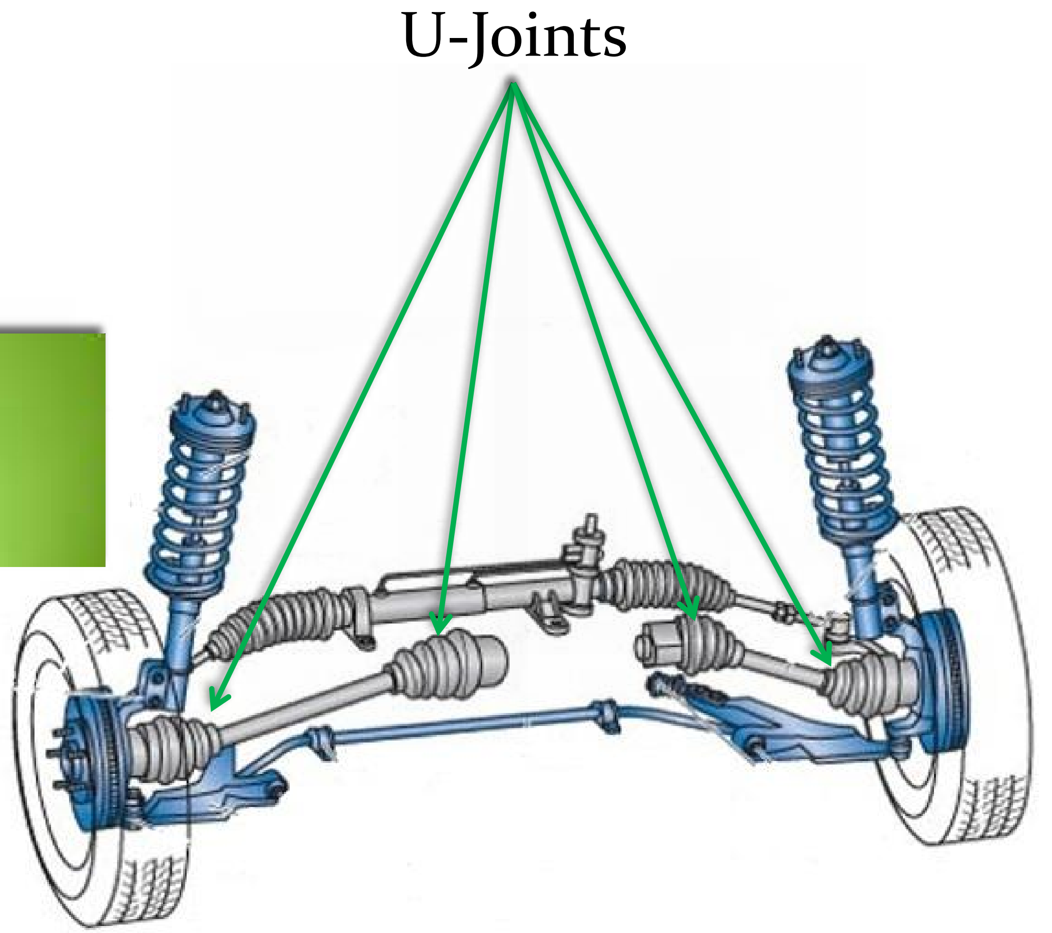
Common Applications

Vehicle Steering Systems

Transmission Line

Front/Rear Driveline

Heavy Duty Vehicles

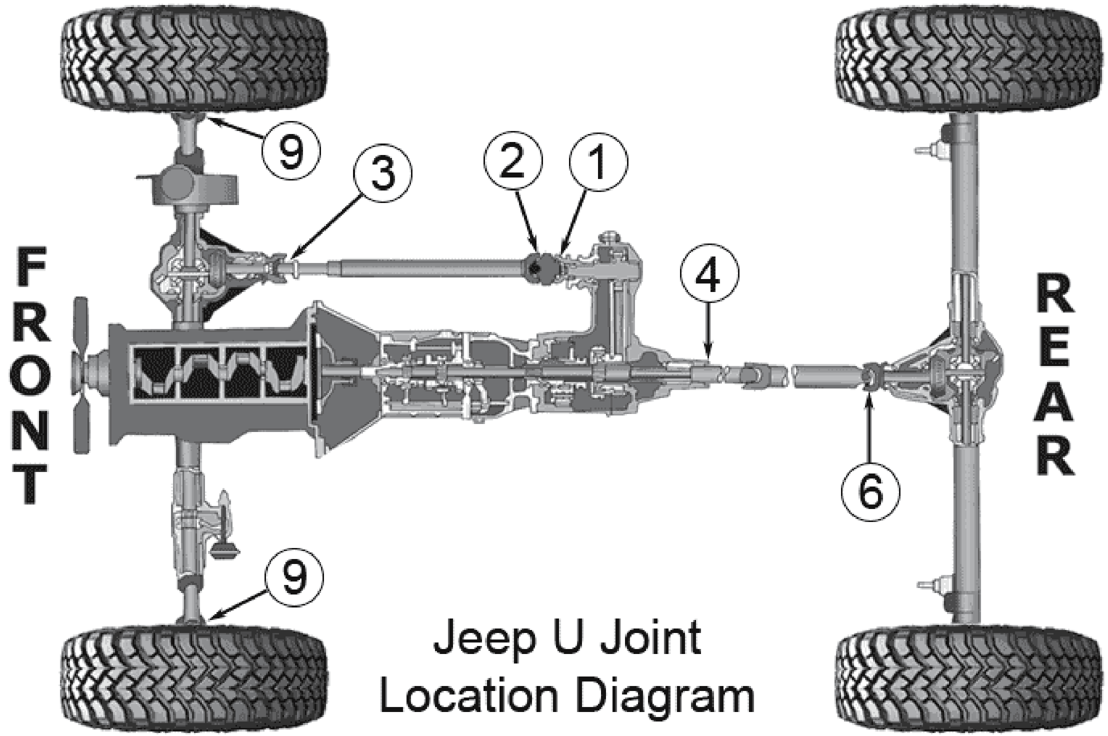


Common Applications

Vehicle Steering Systems

Transmission Line

Front/Rear
Driveline



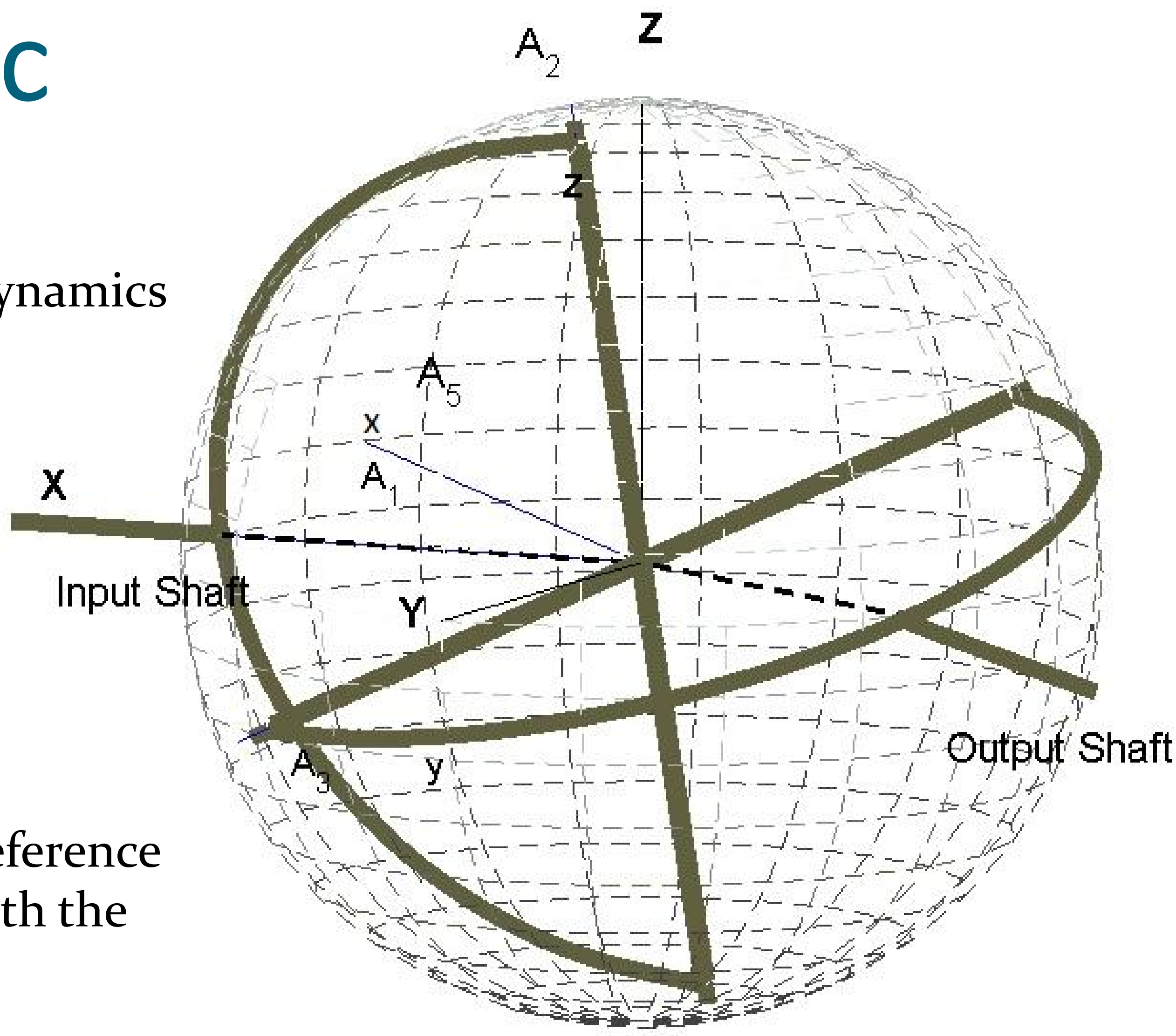
Heavy Duty Vehicles

Schematic

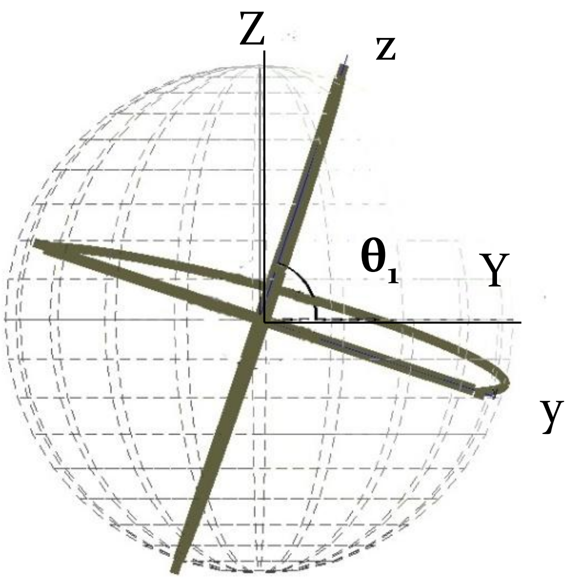
- In order to study the kinematics and the dynamics of the joint, spherical geometry is used

- **XYZ** is the fixed reference frame

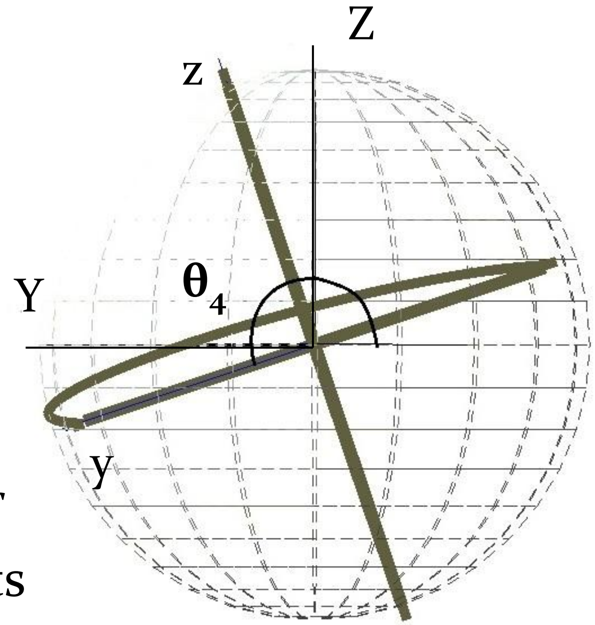
- **xyz** is the moving reference frame, consistent with the spider



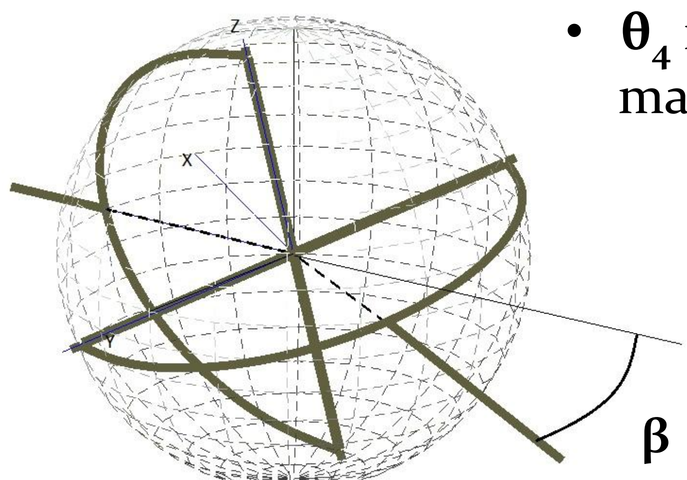
Nomenclature



- β is the angle between the axes of the connected shafts



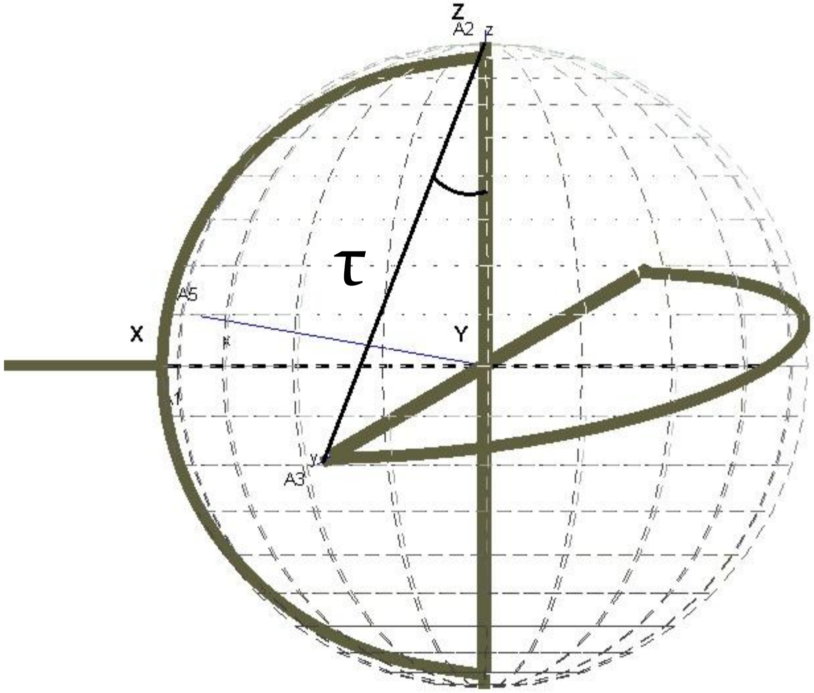
- θ_1 is the angle that A_2 makes with the XY plane



- θ_4 is the angle that A_3 makes with the XY plane

Nomenclature

- $\mathbf{I_{xx}}$, $\mathbf{I_{yy}}$, $\mathbf{I_{zz}}$ are the mass moments of inertia of the floating link about x, y and z axes
- $\mathbf{M_x}$, $\mathbf{M_y}$, $\mathbf{M_z}$ are the torque components acting on the floating link along x, y and z axes
- $\boldsymbol{\tau}$ is the angle between YZ-plane and plane A_2A_3O
- $\boldsymbol{\varepsilon} = 1 - (I_{yy}/I_{zz})$, usually $\varepsilon = 0$
- $\boldsymbol{\lambda} = I_{xx}/2\, I_{zz}$
- **T** is the static torque
- $J = (I_{yy} + I_{zz} - I_{xx})/I_{zz} = 2(1 - \lambda) - \varepsilon$



Nomenclature

- $\mathbf{T_{1H}, T_{1V}}$ are the horizontal and vertical rocking torques on input shaft bearings
- $\mathbf{T_{4H}, T_{4V}}$ are the horizontal and vertical rocking torques on output shaft bearings
- $\mathbf{T_x, T_y, T_z}$ are the inertia torque components of floating link along x,y and z axes
- $\mathbf{T_X, T_Y, T_Z}$ are the inertia torque components of floating link along X,Y and Z axes
- $\boldsymbol{\omega}$ is the angular velocity of input shaft, assumed constant
- $\boldsymbol{\omega_x, \omega_y, \omega_z}$ are the angular velocity components of floating link along x-y-z axes, respectively

Kinematics

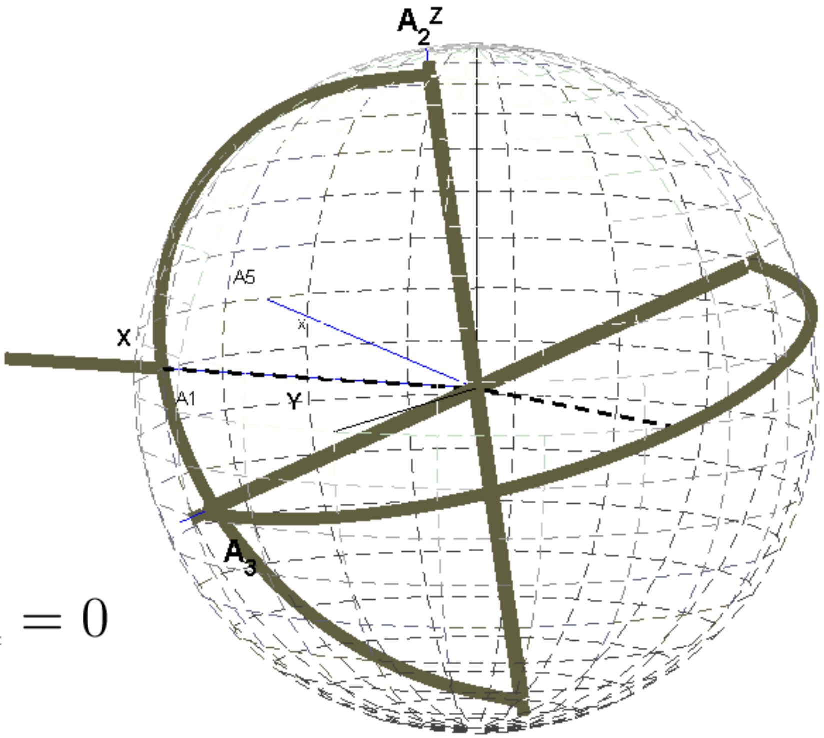
$$A_2 = (0, \cos\theta_1, \sin\theta_1)$$
$$A_3 = (-\sin\beta\cos\theta_4, -\cos\beta\cos\theta_4, \sin\theta_4)$$

y and *z* axes must be always perpendicular

$$A_2 \cdot A_3 = 0$$

$$A_2 \cdot A_3 = -\cos\beta\cos\theta_1\cos\theta_4 + \sin\theta_1\sin\theta_4 = 0$$

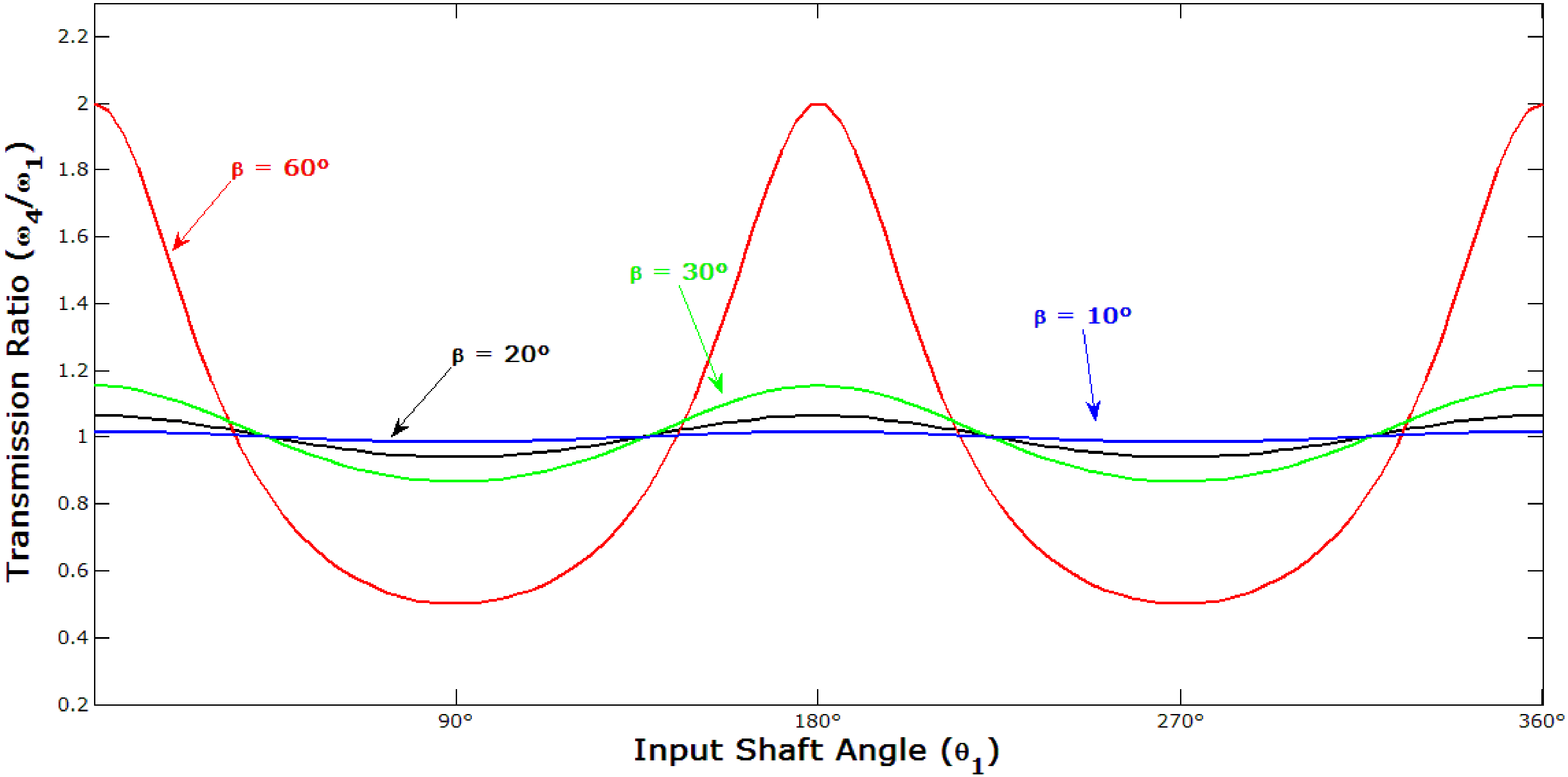
$$\tan\theta_1 \tan\theta_4 = \cos\beta$$



Differentiating by time it is possible to determine the tranmission ratio:

$$|\eta| = \frac{\cos\beta(1 + \tan^2\theta_1)}{\tan^2\theta_1 + \cos^2\beta}$$

Transmission Ratio



Inertia Torque Analysis

- It is convenient to express the moment of inertia of the floating link as follow:

$$I_{xx} = 2\lambda I$$
$$I_{yy} = (1-\epsilon)I$$
$$I_{zz} = I$$

Where λ and ε are constants

- To find the inertia torque Euler's moment equations for motion with a fixed point is used, the axes being principal axes:

$$M_x = I_{xx}\dot{\omega}_x + \omega_y\omega_z(I_{zz} - I_{yy})$$
$$M_y = I_{yy}\dot{\omega}_y + \omega_x\omega_z(I_{xx} - I_{zz})$$
$$M_z = I_{zz}\dot{\omega}_z + \omega_x\omega_y(I_{yy} - I_{xx})$$

Inertia Torque Analysis

- Combining the two sets of equations it is possible to get:

$$M_x = I\omega^2 u(\theta_1)$$
$$M_y = I\omega^2 v(\theta_1)$$
$$M_z = I\omega^2 w(\theta_1)$$

Where *I* and *ω* are constant and the three functions $u\ (\theta_1)$, $v\ (\theta_1)$, $w\ (\theta_1)$ are the following:

$$u(\theta_) = \frac{2\lambda\dot{\omega}_x + \epsilon\omega_y\omega_z}{\omega^2}$$
$$v(\theta_1) = \frac{\dot{\omega}_y + (2\lambda - 1)\omega_x\omega_z - \epsilon\dot{\omega}_y}{\omega^2}$$
$$w(\theta_1) = \frac{\dot{\omega}_z - (2\lambda - 1)\omega_x\omega_y - \epsilon\omega_x\omega_y}{\omega^2}$$

Inertia Torque Analysis

- The angular velocity components ω_x and ω_y can be easily determined as:

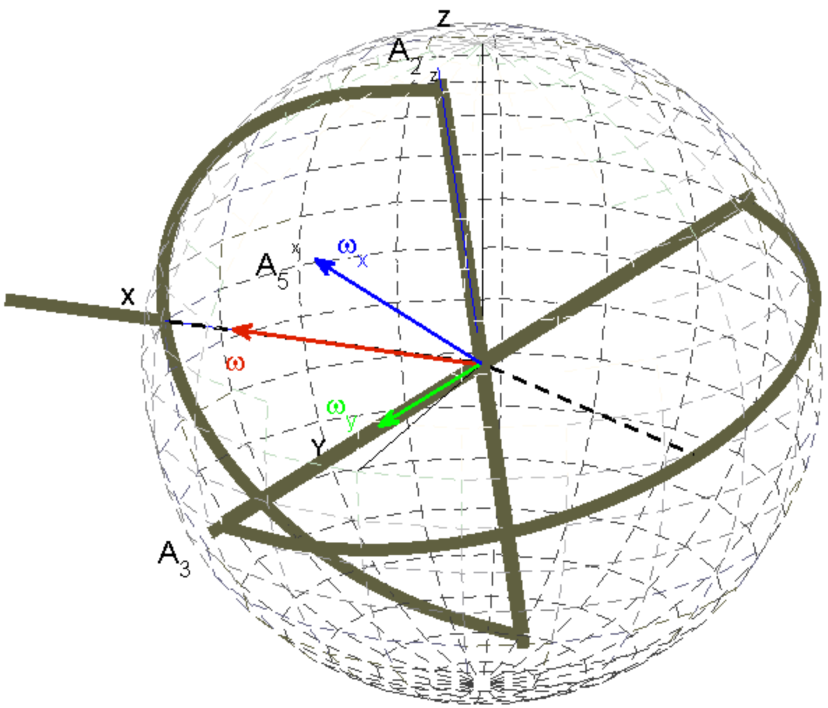
$$\omega_x = \omega \cos \widehat{A_1 0 A_5}$$
$$\omega_y = \omega \cos \widehat{A_1 0 A_3}$$

Where:

$$A_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} -\sin\beta\cos\theta_4 \\ -\cos\beta\cos\theta_4 \\ \sin\theta_4 \end{pmatrix}$$

And A_5 can be evaluated as:

$$A_5 = A_3 \times A_2$$



Inertia Torque Analysis

- This gives:

$$A_5 = \begin{pmatrix} -\cos\beta\sin\theta_1\cos\theta_4 - \cos\theta_1\sin\theta_4 \\ \sin\beta\sin\theta_1\cos\theta_4 \\ -\sin\beta\cos\theta_1\cos\theta_4 \end{pmatrix}$$

Which in turn yields:

$$\cos\widehat{A_1 0 A_5} = A_1 \cdot A_5 = -\cos\beta\sin\theta_1\cos\theta_4 - \cos\theta_1\sin\theta_4$$
$$\cos\widehat{A_1 0 A_3} = A_1 \cdot A_3 = -\sin\beta\cos\theta_4$$

And hence:

$$\omega_x = -\omega(\cos\beta\sin\theta_1\cos\theta_4 + \cos\theta_1\sin\theta_4)$$
$$\omega_y = -\omega\sin\beta\cos\theta_4$$

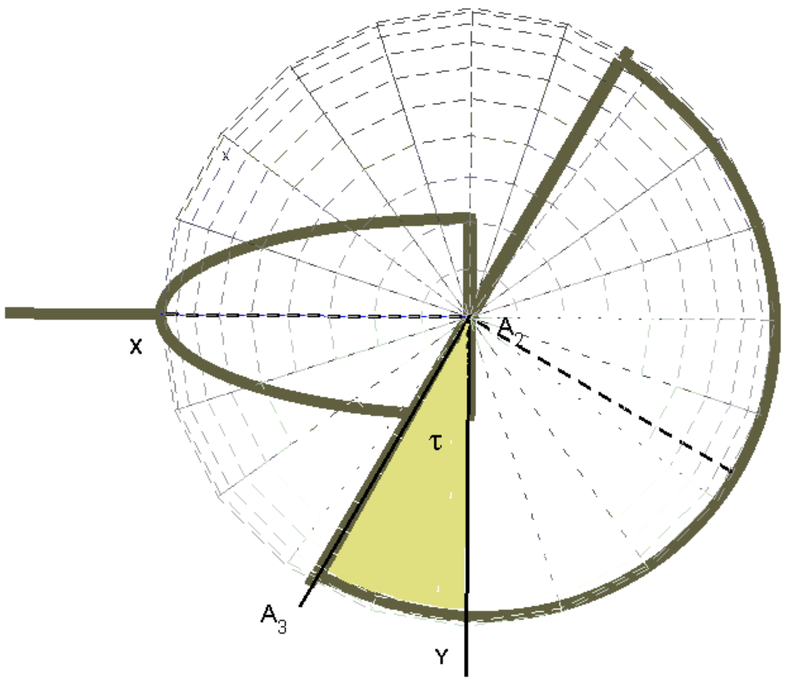
Inertia Torque Analysis

ω_z can be found considering that:

$$\omega_z = -\dot{\tau}$$

Where *sin* τ is determined by geometry :

$$\sin\tau = -\sin\beta\cos\theta_4$$
$$\tau = \arcsin(-\sin\beta\cos\theta_4)$$



Differentiating by time:

$$-\dot{\tau} = \omega_z = -\omega\eta\sec\tau\sin\beta\sin\theta_4$$

Inertia Torque Analysis

- Differentiating by time it is possible to get the expressions of ω_x, ω_y, and ω_z. After some algebraic passages the following expressions are obtained:

$$\dot{\omega}_x = -\eta\omega^2(\cos\theta_1\cos\theta_4 - \cos\beta\sin\theta_1\sin\theta_4)$$
$$\dot{\omega}_y = \eta\omega^2\sin\beta\sin\theta_4$$
$$\dot{\omega}_z = \omega^2\sec\tau\sin\beta(\eta^2(\sin^2\beta\sin^2\theta_4\sec^2\tau\cos\theta_4 - 1) + \frac{\dot{\eta}}{\omega})$$

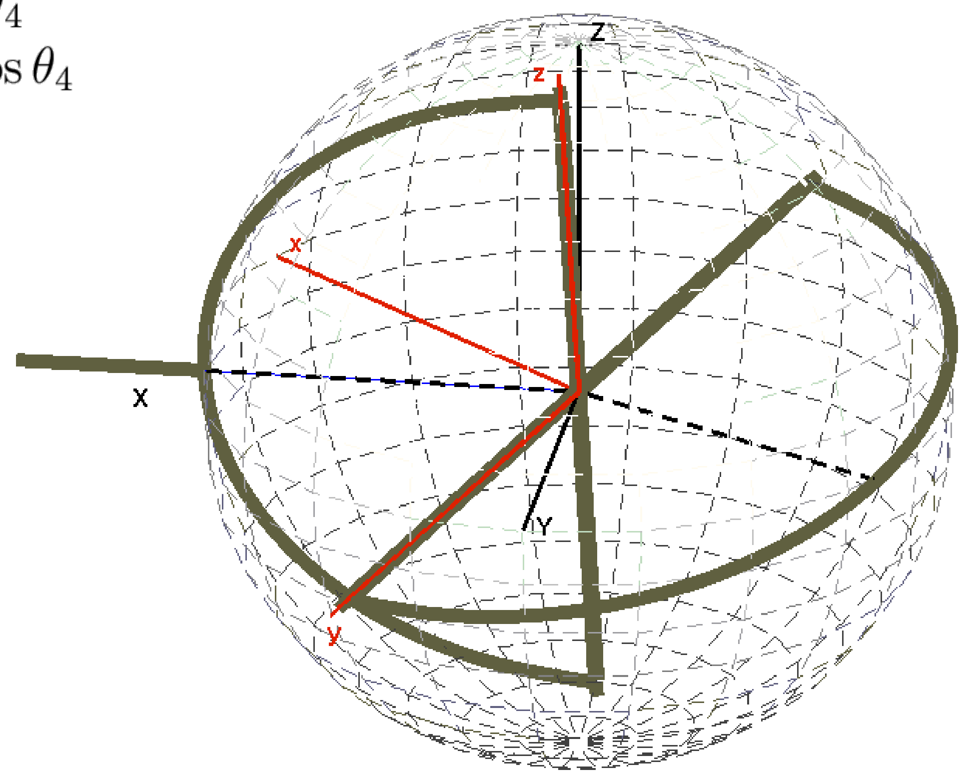
In fixed-system coordinates, the inertia torque components are given by the equations:

$$T_X = -M_x\cos(x, X) - M_y\cos(y, X) - M_z\cos(z, X)$$
$$T_Y = -M_x\cos(x, Y) - M_y\cos(y, Y) - M_z\cos(z, Y)$$
$$T_Z = -M_x\cos(x, Z) - M_y\cos(y, Z) - M_z\cos(z, Z)$$

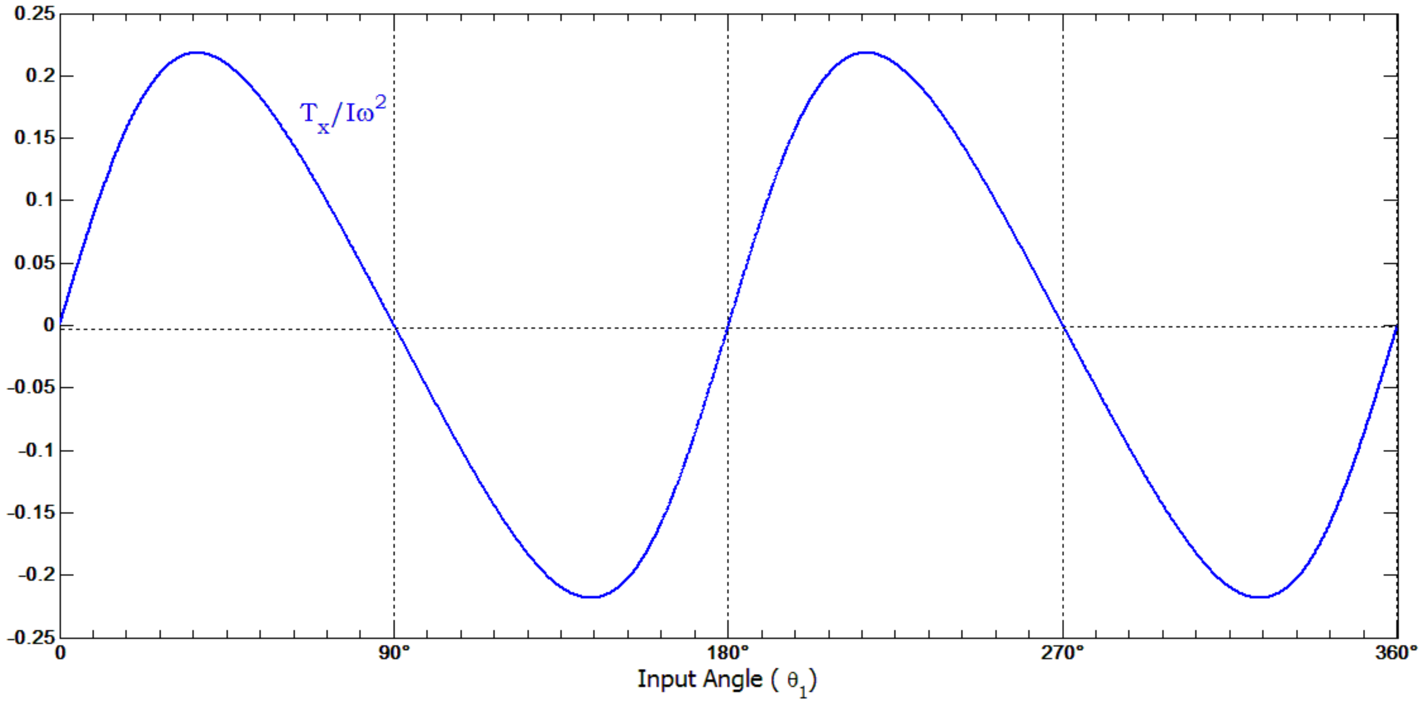
Inertia Torque Analysis

- Where:

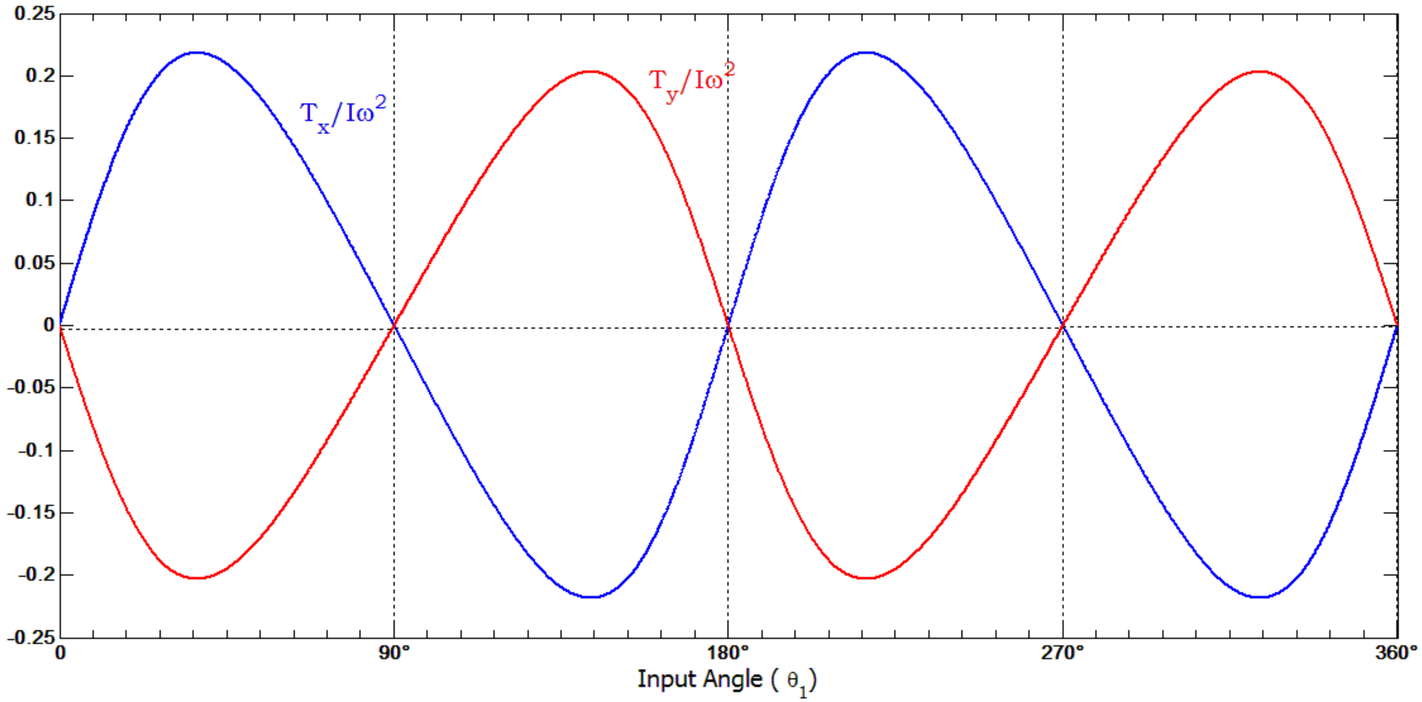
$$
\begin{aligned}
\cos(x, X) &= -(\cos\beta\cos\theta_4\sin\theta_1 + \cos\theta_1\sin\theta_4) \\
\cos(x, Y) &= \sin\beta\sin\theta_1\cos\theta_4 \\
\cos(x, Z) &= -\sin\beta\cos\theta_1\cos\theta_4 \\
\cos(y, X) &= -\sin\beta\cos\theta_4 \\
cos(y, Y) &= -\cos\beta\cos\theta_4 \\
\cos(y, Z) &= \sin\theta_4 \\
\cos(z, X) &= 0 \\
\cos(z, Y) &= \cos\theta_1 \\
\cos(z, Z) &= \sin\theta_1
\end{aligned}
$$



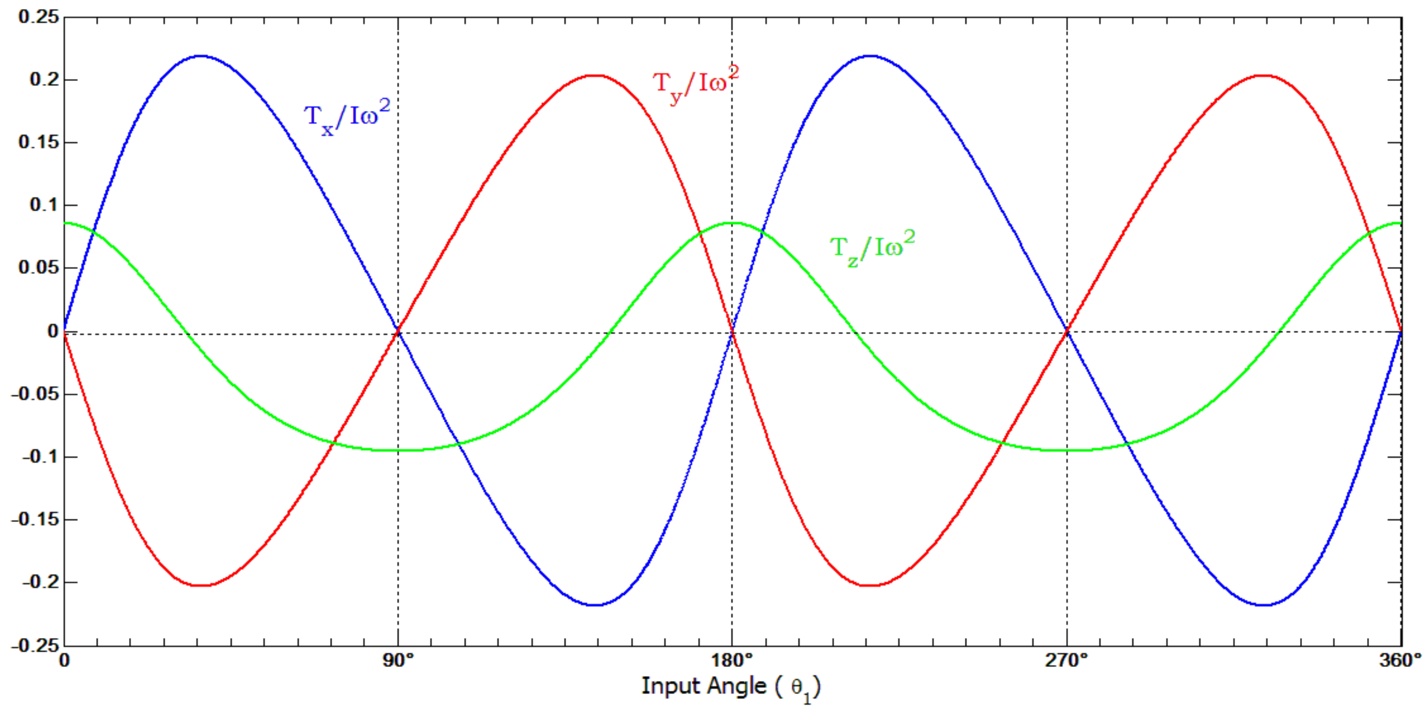
Inertia Torque Analysis



Inertia Torque Analysis



Inertia Torque Analysis

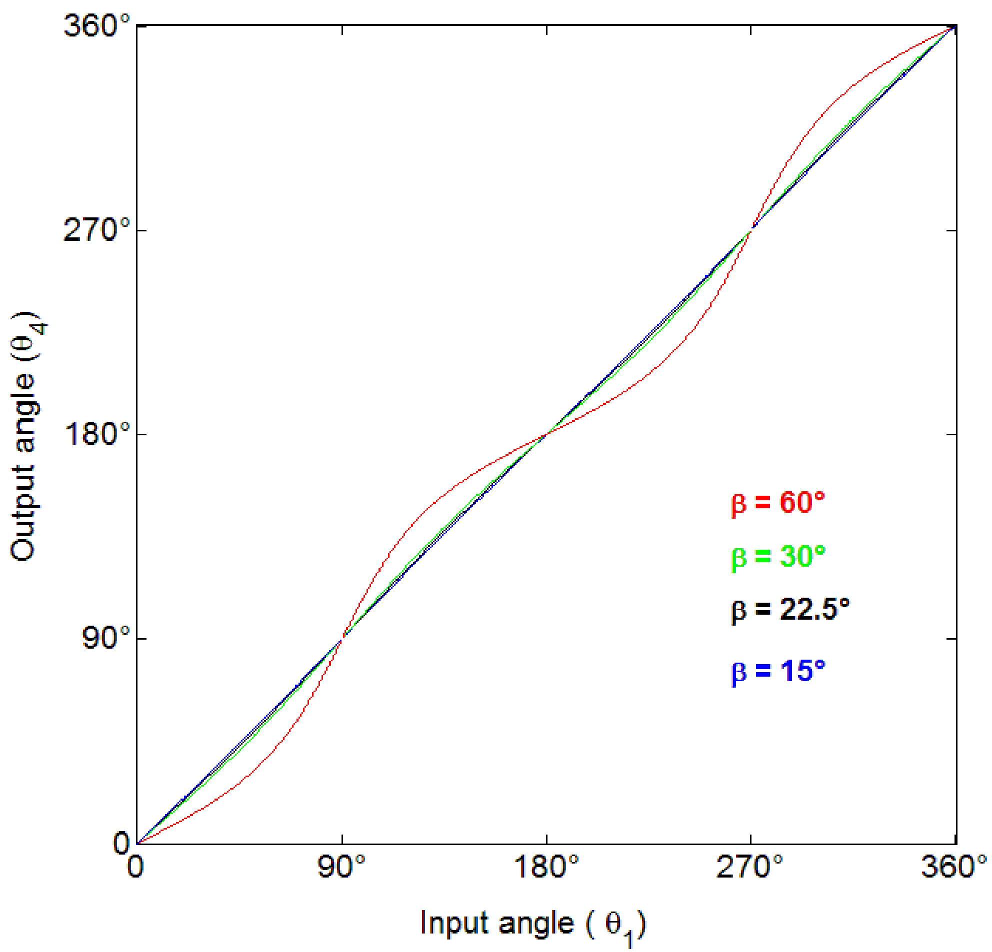


Approximated Equations

An approximated solution can be evaluated doing the following assumptions:

$$\sin\beta \simeq \beta$$
$$\cos\beta \simeq 1$$
$$\theta_4 \simeq \theta_1$$

These seem to be reasonable since rarely angle β exceeds 30°, especially in high-speed and high-load applications, where it is generally much smaller



Approximated Equations

- With these assumptions the approximate equations for the inertia torques with terms of first, second and third order in β, are the following, being $J = (I_{yy} + I_{zz} - I_{xx})/I_{zz} = 2\,(1 - \lambda) - \varepsilon$:

$$\frac{T_X}{I\omega^2} = (1 - J)\beta^2 \sin 2\theta_1$$

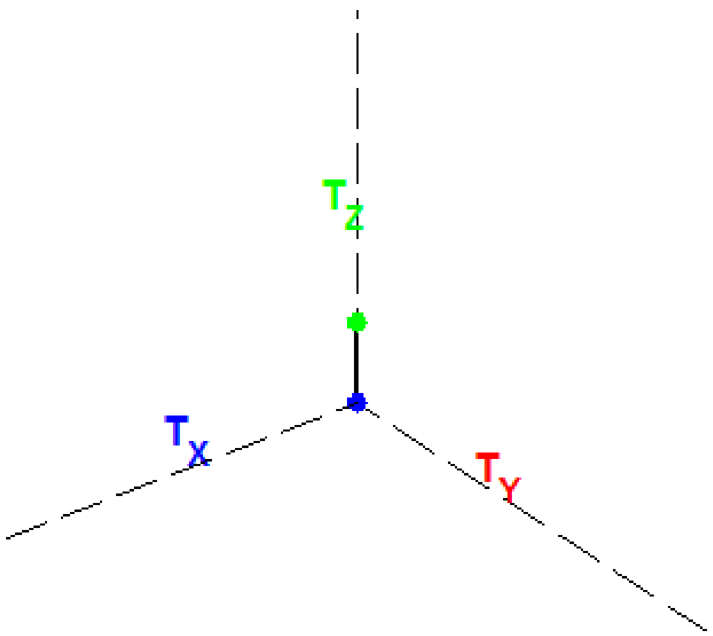
$$\frac{T_Y}{I\omega^2} = -J\beta \sin 2\theta_1 + \frac{1}{3}\beta^3(2J - 3)\sin 2\theta_1 - \frac{1}{2}\beta^3 J \sin 4\theta_1$$

$$\frac{T_Z}{I\omega^2} = J\beta \cos 2\theta_1 - \frac{1}{6}\beta^3 J \cos 2\theta_1 + \frac{1}{2}\beta^3 J \cos 4\theta_1$$

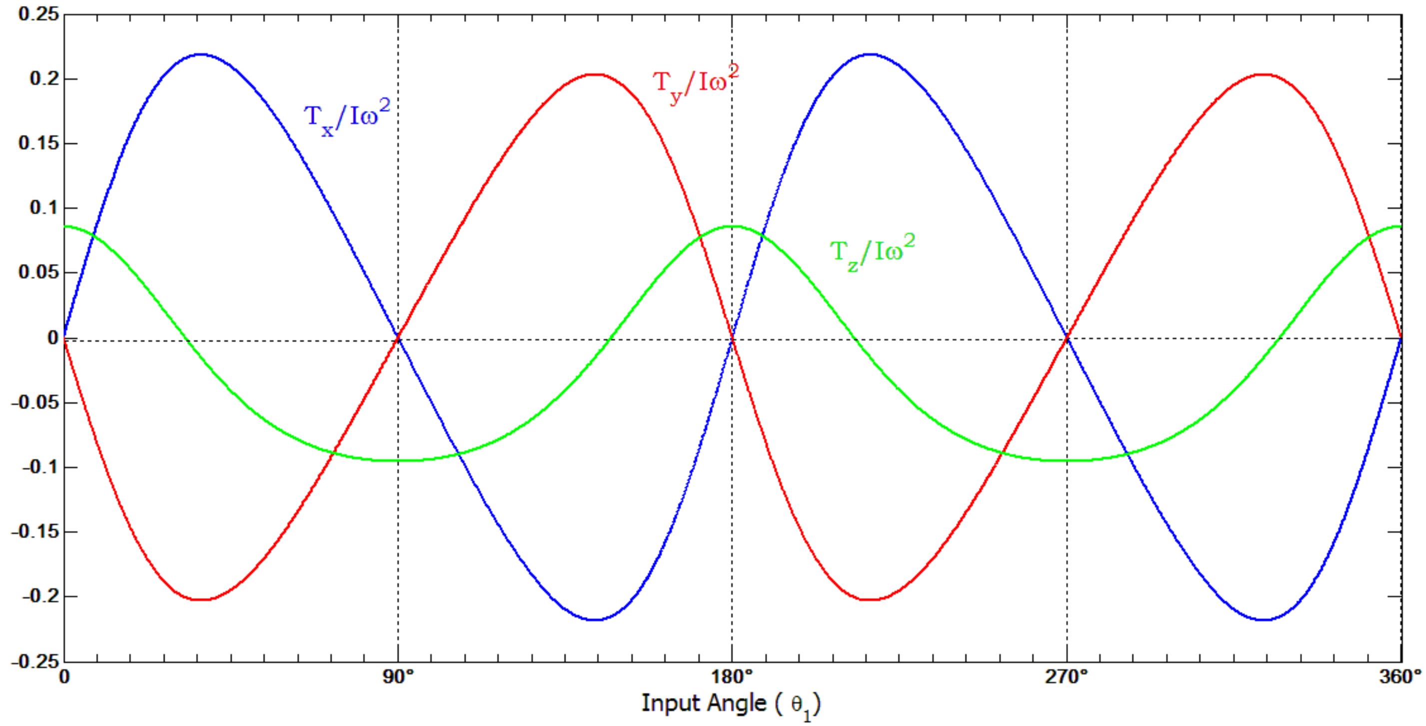
Notice that all the functions have a period of π. Furthermore, the vector sum of the first-order torque components of T_Y and T_Z constitute a single rotating vector having twice input speed and phased 90° from the plane of the input torque.

Approximated Equations

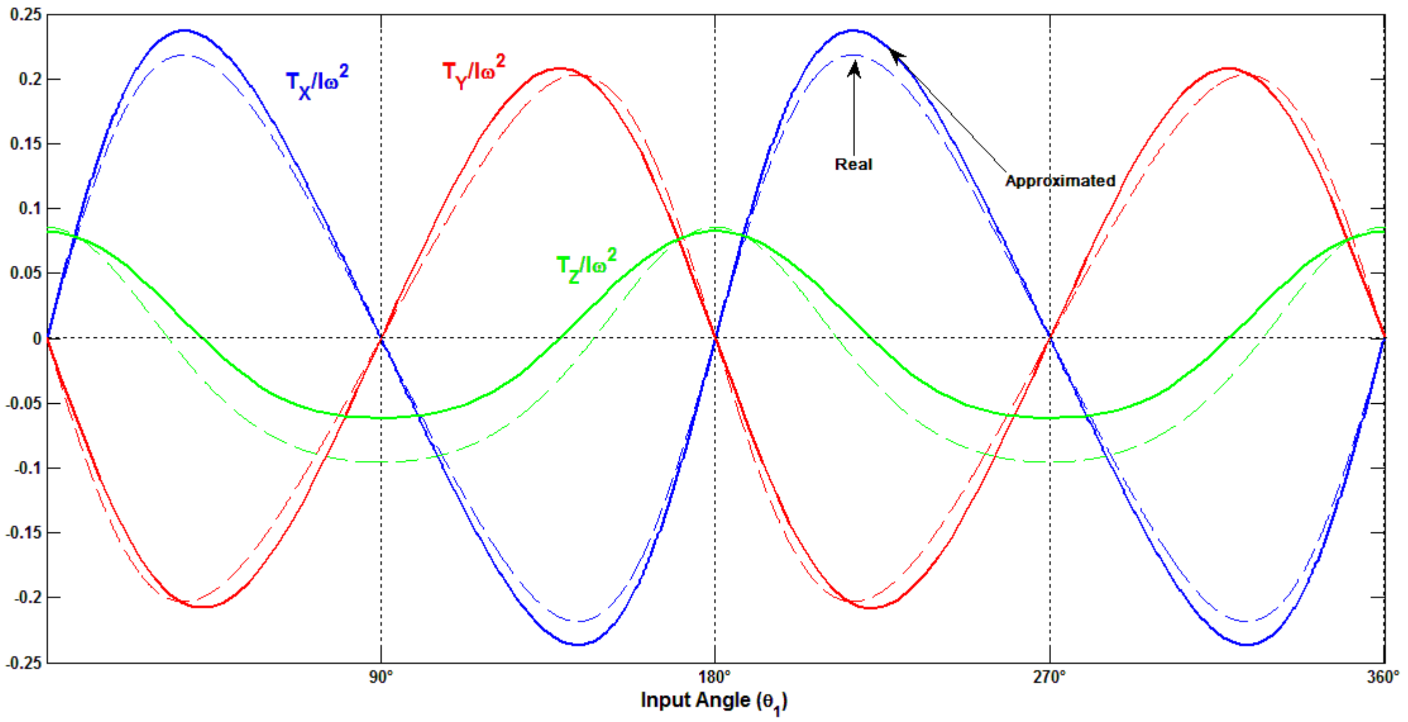
$$\frac{T_Y + iT_Z}{I\omega^2} = J\beta e^{i(2\theta_1 + \frac{\pi}{2})} + o(\beta^3)$$



Approximated Equtions

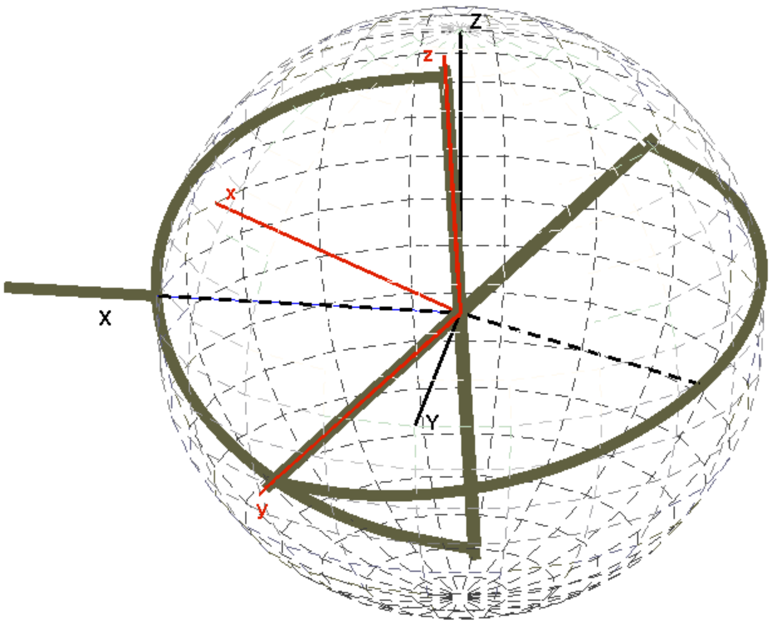


Approximated Equations



Rocking Torque Analysis

- In the absence of friction, a pin joint cannot transmit moments about its axis. Hence, the inertia component (- M_z) is transmitted only to the output shaft, while the component (- M_y) is transmitted only to the input shaft.
- The torque component (- Mx) is trasmitted toi both shafts, but in different proportions, let's say a fraction γ to the output and a fraction $(1 - \gamma)$ to the input.



- The determination of γ is a difficult task and cannot be determined from rigid-body considerations alone.

Rocking Torque Analysis

- Since the inertia of the output shaft is not considered, an approximate, constant value of γ can be determined from the condition that there will be no output torque due to the inertia torques of the floating link :

$$\gamma M_x \cos \widehat{A_5 0 A_4} + M_z \cos \widehat{A_2 0 A_4} = 0$$

Where in the chosen model A_4 is the intersection between the output shaft direction and the sphere:

$$A_4 = (-\cos\beta, \sin\beta, 0)$$
$$\cos \widehat{A_5 0 A_4} = A_5 \cdot A_4 = \sin\theta_1 \cos\theta_4 + \cos\beta \cos\theta_1 \sin\theta_4$$
$$\cos \widehat{A_2 0 A_4} = A_2 \cdot A_4 = \cos\theta_1 \sin\beta$$

Rocking Torque Analysis

- Introducing the parameter J and using the approximated equations, it is possible to obtain a constant value for γ, thus not depending on the input angle:

$$J = (I_{yy} + I_{zz} - I_{xx})/I_{zz} = 2(1-\lambda) - \epsilon$$

$$\gamma M_x \cos \widehat{A_5 0 A_4} + M_z \cos \widehat{A_2 0 A_4} = 0$$

$$\gamma \simeq \frac{-J}{2-J}$$

On the other hand the fraction transmitted to the input is:

$$1 - \gamma = \frac{2}{2-J}$$

Rocking Torque Analysis

- The horizontal and vertical rocking torques transmitted to the input shaft can be expressed as:

$$T_{1H} = \vec{M}_y \cdot \vec{j} + (1-\gamma)\vec{M}_x \cdot \vec{j}$$
$$T_{1H} = -v(\theta_1)\cos\beta\cos\theta_4 + (1-\gamma)u(\theta_1)\sin\beta\cos\theta_4\sin\theta_1$$

$$T_{1V} = \vec{M}_y \cdot \vec{k} + (1-\gamma)\vec{M}_x \cdot \vec{k}$$
$$T_{1V} = v(\theta_1)\sin\theta_4 - (1-\gamma)u(\theta_1)\sin\beta\cos\theta_4\cos\theta_1$$

Where $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors along the fixed X-Y-Z axes.

Using the approximate equations for inertia torques the resuls are:

$$\frac{T_{1H}}{I\omega^2} = \frac{1}{2}\beta J \sin 2\theta_1$$
$$\frac{T_{1V}}{I\omega^2} = -\frac{1}{2}\beta J(1+\cos 2\theta_1)$$

Rocking Torque Analysis

- Similarly the rocking torques on the output shaft can be estimated:

$$T_{4H} = \vec{M}_z \cdot \vec{t} \cos\beta + \gamma \vec{M}_x \cdot \vec{t} \cos\beta$$
$$T_{4H} = w \cos\beta \cos\theta_1 - \gamma u(\theta_1) \sin\beta \cos\theta_1 \sin\theta_4$$

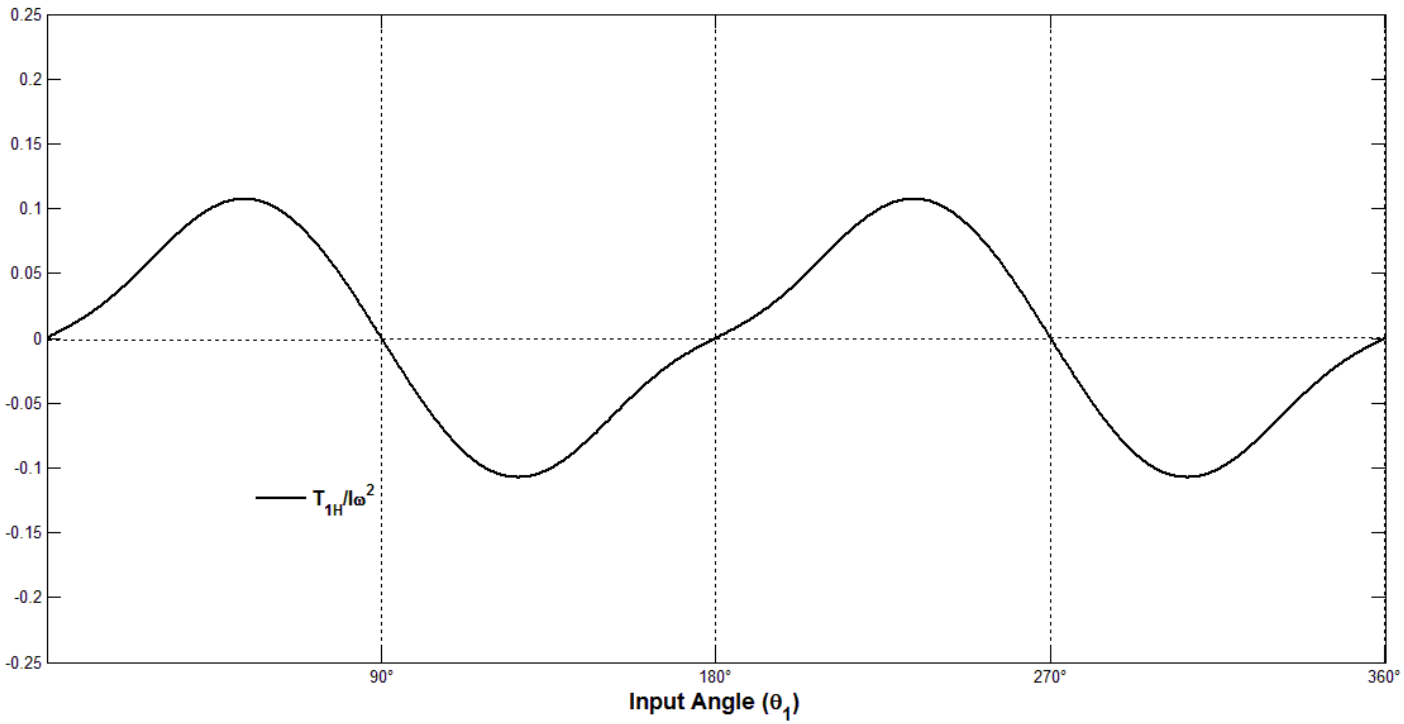
$$T_{4V} = \vec{M}_z \cdot \vec{k} + \gamma \vec{M}_x \cdot \vec{k}$$
$$T_{1V} = w(\theta_1) \sin\theta_1 - \gamma u(\theta_1) \sin\beta \cos\theta_4 \cos\theta_1$$

Where $\vec{t}$ is the vector $\vec{t} = (\sin\beta, \cos\beta, 0)$

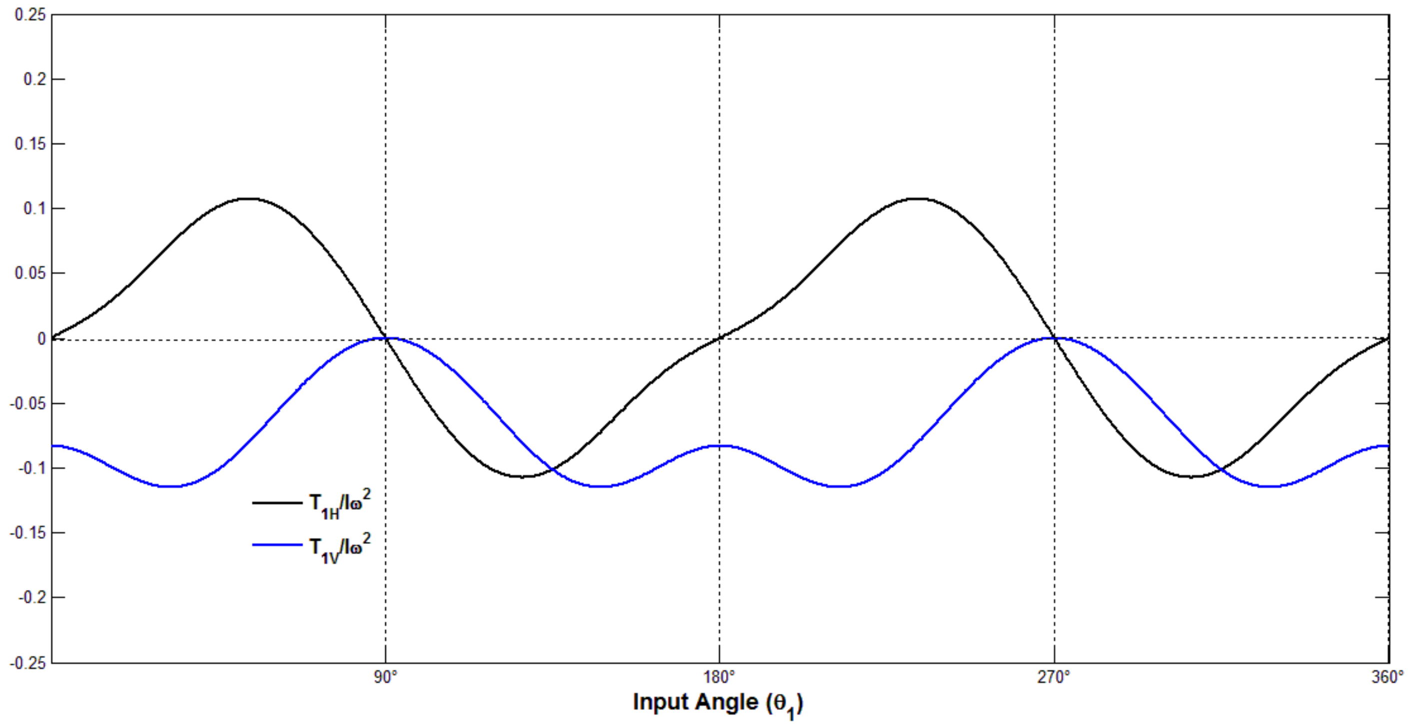
Using the approxmate equations for inertia torques the results are:

$$\frac{T_{4H}}{I\omega^2} = \frac{1}{2}\beta J \sin 2\theta_1$$
$$\frac{T_{4V}}{I\omega^2} = -\frac{1}{2}\beta J (1 - \cos 2\theta_1)$$

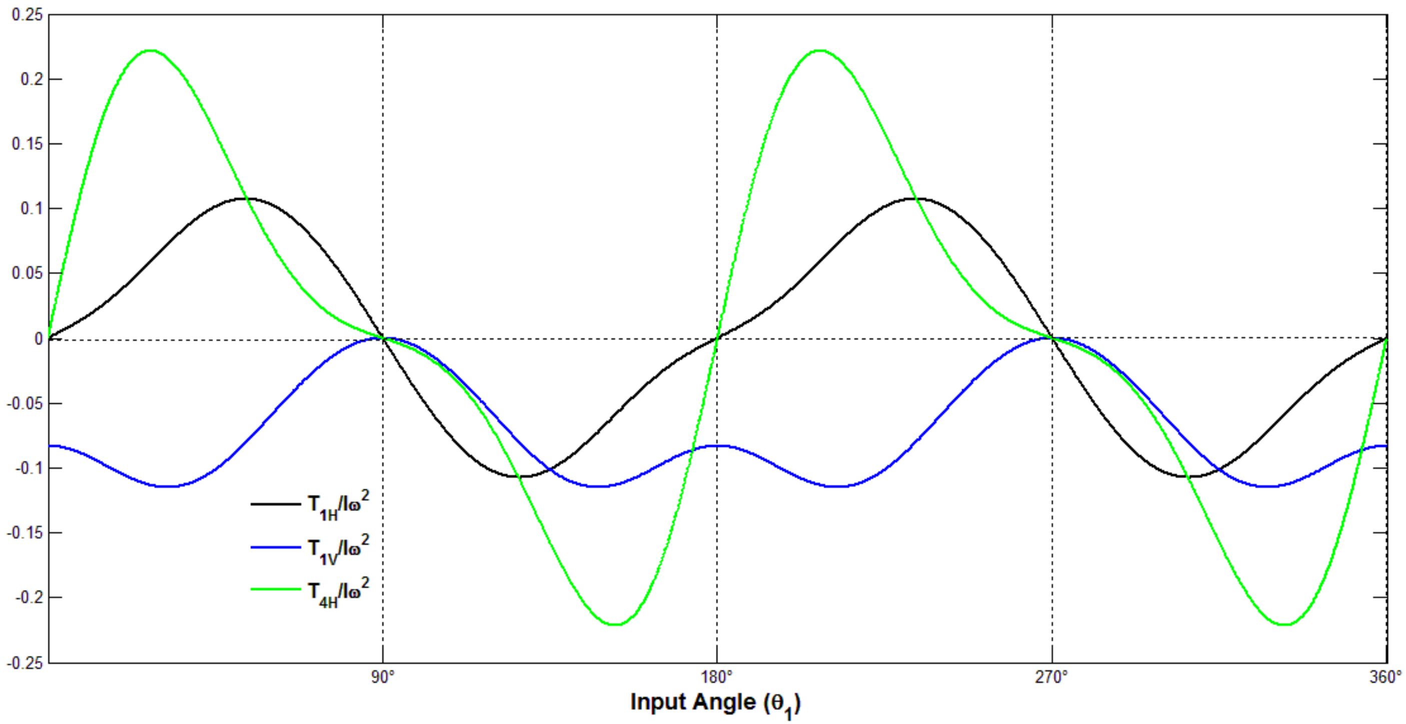
Rocking Torque Analysis



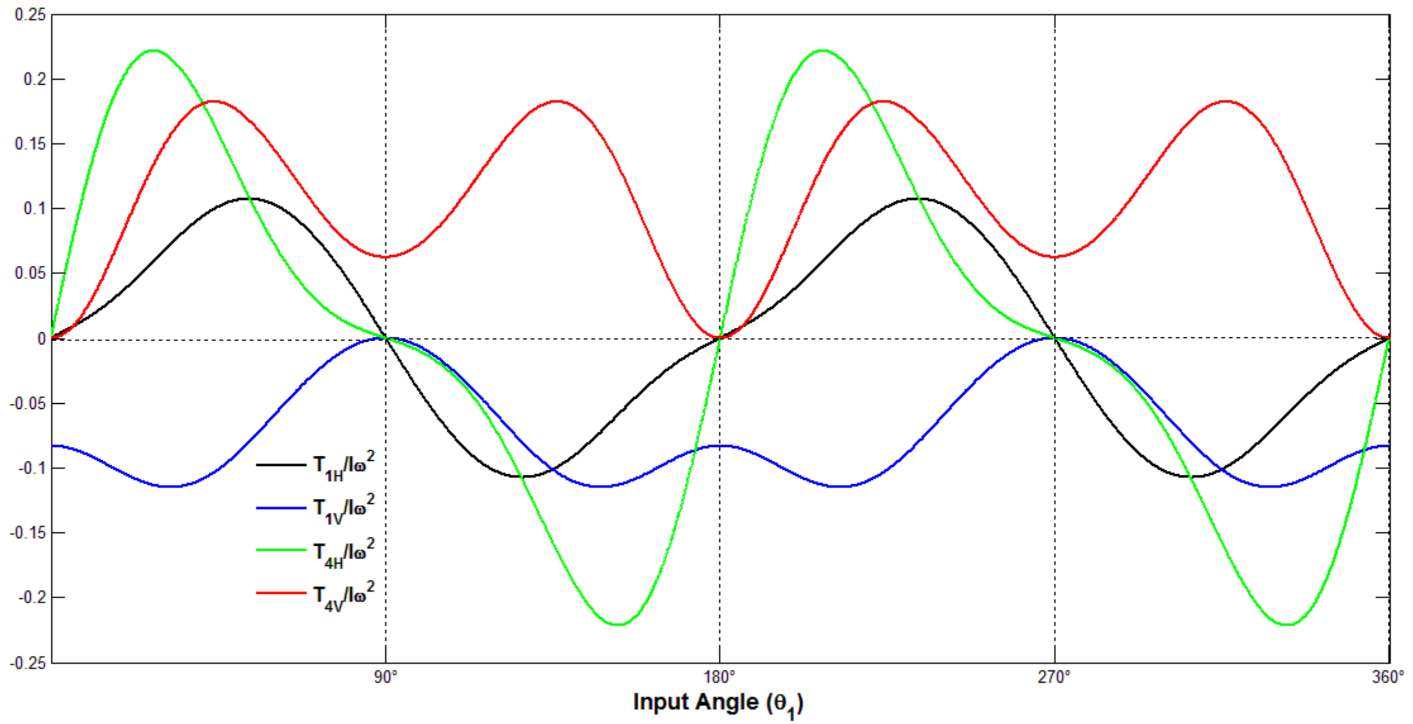
Rocking Torque Analysis



Rocking Torque Analysis



Rocking Torque Analysis

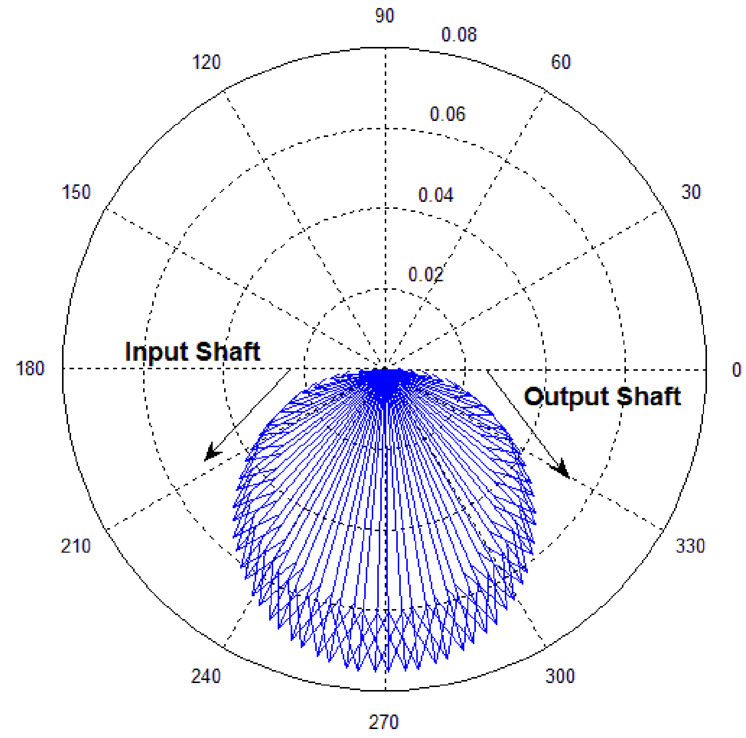


Rocking Torque Analysis

- The vector sum of the horizontal and vertical components for both input and output shafts yields a steady torque and a rotating component rotating at twice input speed and leding the plane of the input fork of 90°.

$$T_1 = \frac{1}{2}\beta J I \omega^2 e^{\frac{3}{2}\pi i} + \frac{1}{2}\beta J I \omega^2 e^{i(2\theta_1 + \frac{3}{2}\pi)}$$

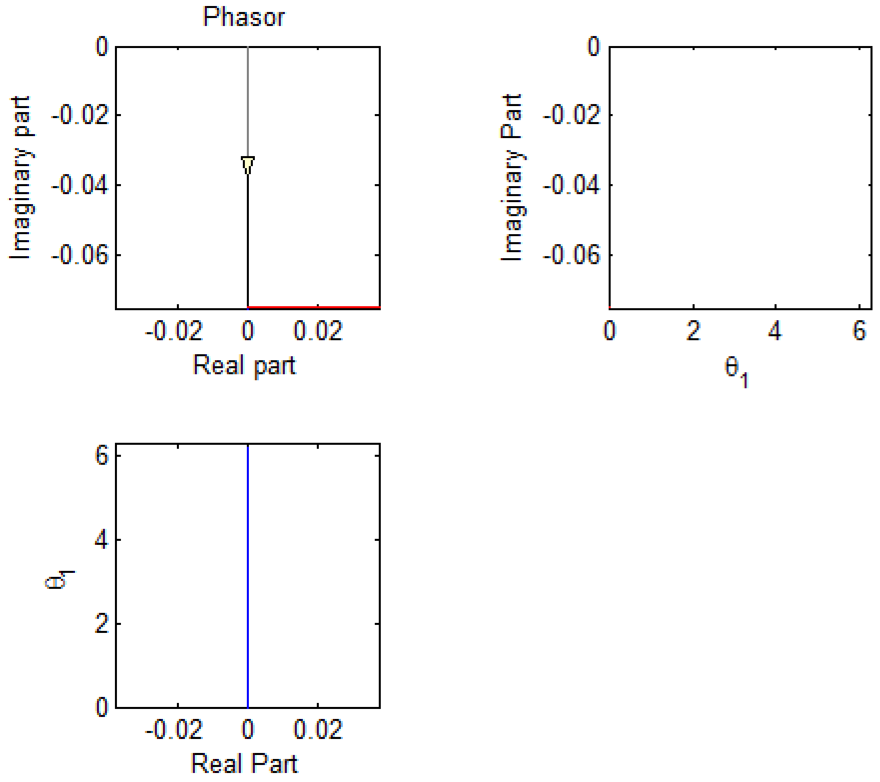
$$T_4 = \frac{1}{2}\beta J I \omega^2 e^{\frac{3}{2}\pi i} + \frac{1}{2}\beta J I \omega^2 e^{-i(2\theta_1 + \frac{3}{2}\pi)}$$



The difference between input and output rocking torque is just in the direction of rotation of the varying vector

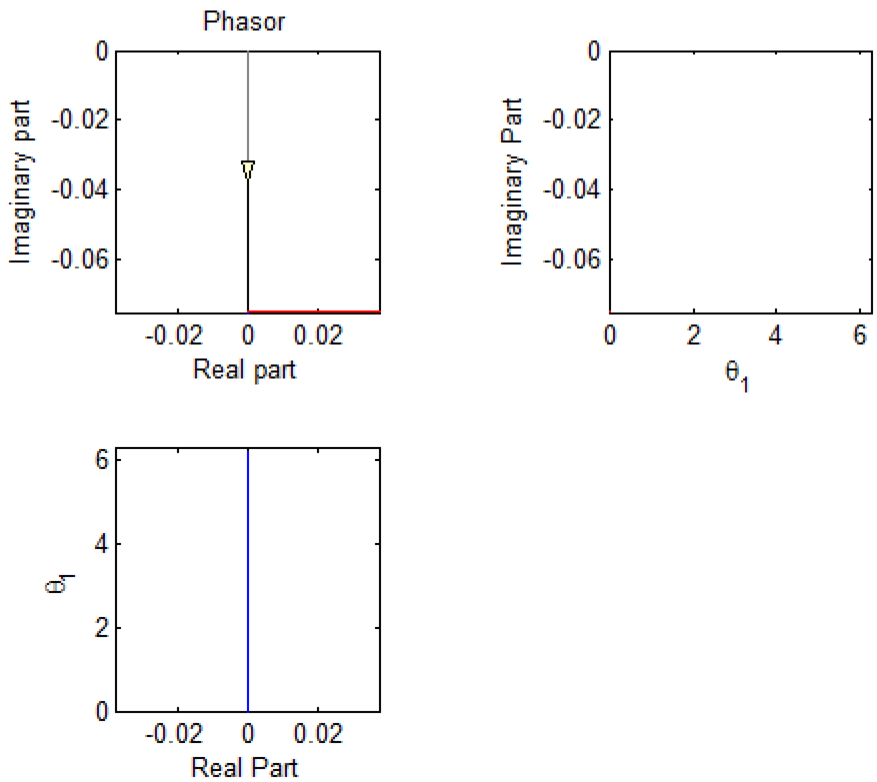
Rocking Torque Analysis

- Input Shaft Rocking Torque

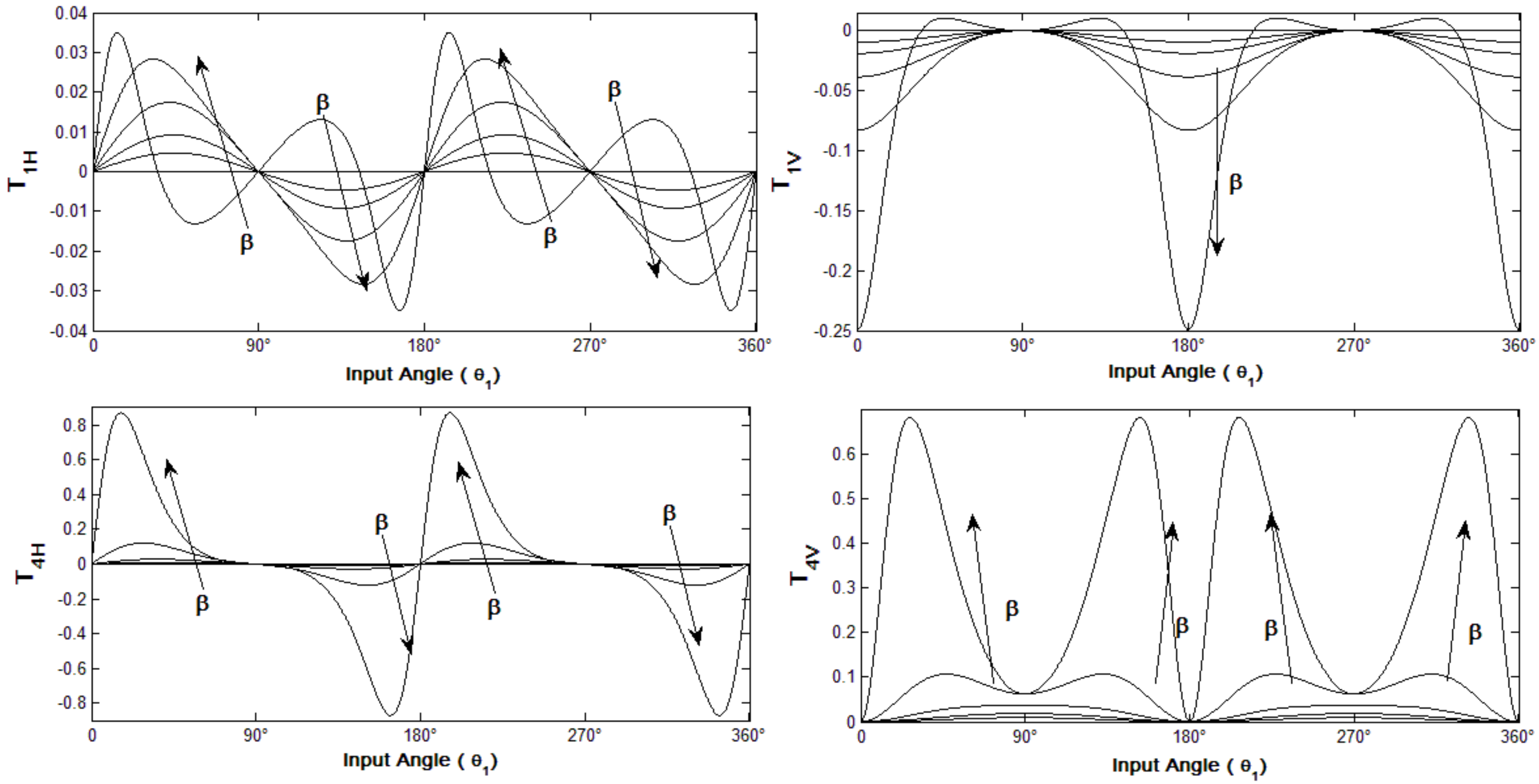


Rocking Torque Analysis

- Output Shaft Rocking Torque



Rocking Torque Analysis



Conclusions

- The inertia torques are of even order, the predominant harmonic being of order 2
- The inertia torque component along the direction of the input shaft axis is of order β^2. The other two components are of order β
- The dominant terms of the inertia torque components, T_Y and T_Z, vanish if the mass distribution were such that J vanishes. This can be achieved by locating the mass of the floating link as close as possible to the plane of the pin joints
- The approximate solutions are precise for common values of β and allow a deep insight into the dynamics of the system
- The approximate solutions for the rocking torques provide a steady torque and a rotating component, equal in magnitude for both input and output shaft

Conclusions

- The magnitude of the statically induced rocking torques on either shaft is approximately $T\beta$, where T denotes the static transmitted torque by the Cardan joint. At critical speed this will be nearly equal to the rocking torques due to the inertia of the floating link; this leads to an estimation of the critical speed:

$$\omega_{CR} = \sqrt{\frac{2T}{IJ}}$$

Thank you for your attention!